AUBRY SETS FOR WEAKLY COUPLED SYSTEMS OF HAMILTON–JACOBI EQUATIONS

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Abstract. We introduce a notion of Aubry set for weakly coupled systems of Hamilton–Jacobi equations on the torus and characterize it as the region where the obstruction to the existence of globally strict critical subsolutions concentrates. As in the case of a single equation, we prove the existence of critical subsolutions which are strict and smooth outside the Aubry set. This allows us to derive in a neat way a comparison result among critical sub and supersolutions with respect to their boundary data on the Aubry set, showing in particular that the latter is a uniqueness set for the critical system. Furthermore, we show that the trace of any critical subsolution on this set can be extended to the whole torus in such a way that the output is a critical solution. We also highlight some rigidity phenomena taking place on the Aubry set: first, the values taken by the differences of the components of a critical subsolution, on this set, are independent of the specific subsolution chosen; second, for each point y in the Aubry set, there exists a vector which is a reachable gradient at y of any critical subsolution.

Introduction

In the study of the Hamilton–Jacobi equation for Tonelli Hamiltonians, weak KAM theory [14, 15, 16, 18] is a bridge between the PDE *per se*, and tools from the theory of dynamical systems. The use of the Lax–Oleinik formula, which is a variational formula to represent the viscosity solutions of the evolutionary equation, allows one to make rigorous the intuition that a solution of the Hamilton–Jacobi equation

$$\partial_t u + H(x, D_x u) = 0 \quad \text{in } (0, +\infty) \times \mathbb{T}^N \tag{1}$$

is deeply linked to the image of the graph of $D_x u(0, \cdot)$ through the Hamiltonian flow of H. It is therefore rather natural that some invariant and minimizing sets, the Aubry and Mather sets, capture the long time behavior of the evolutionary equation. These sets are included in what can be seen as generalized invariant Lagrangian manifolds. They appear as graphs of differentials of weak KAM (or critical) solutions, that is viscosity solutions of the equation

$$H(x, D_x u) = c \quad \text{in } \mathbb{T}^N, \tag{2}$$

where c is a real number also known as *critical value*.

From the PDE viewpoint the Aubry set turns out to be the region where the obstruction to the existence of globally strict critical subsolutions concentrates. In particular, the existence of a critical subsolution, smooth and strict outside the Aubry set [3, 19, 20], implies that the latter is a uniqueness set for the critical

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equation: two critical solutions agree on the whole torus if they agree on the Aubry set.

The purpose of the present paper is to generalize some PDE aspects of weak KAM theory to the case of weakly coupled systems of first order Hamilton–Jacobi equations. More precisely, we consider a family of systems of the form

$$H_i(x, Du_i) + \sum_{j=1}^{m} b_{ij}(x) u_j(x) = a \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \dots, m\}, \tag{3}$$

where a is a real constant, $H_1, \dots, H_m$ are continuous Hamiltonians defined on the cotangent bundle of $\mathbb{T}^N$, convex and coercive in the momentum variable, and $B(x) := \big(b_{ij}(x)\big)$ is the coupling matrix, i.e. an $m \times m$ matrix with continuous coefficients satisfying

$$b_{ij}(x) \leqslant 0 \quad \text{for } j \neq i, \quad \sum_{j=1}^{m} b_{ij}(x) \geqslant 0 \qquad \text{for every } x \in \mathbb{T}^N \text{ and } i \in \{1, \dots, m\}.$$

The coupling matrix is additionally assumed *irreducible*, meaning, roughly speaking, that the coupling is non–trivial and the system cannot be split into independent subsystems, see Section 1.2 for the precise definition; and *degenerate*, i.e.

$$\sum_{j=1}^{m} b_{ij}(x) = 0 \qquad \text{for every } x \in \mathbb{T}^N \text{ and } i \in \{1, \dots, m\}.$$

Under these assumptions, there exists a unique constant $a \in \mathbb{R}$ for which the system (3) admits viscosity solutions. We characterize such a quantity, hereafter denoted by c and termed *critical value*, as the minimal $a \in \mathbb{R}$ for which the corresponding weakly coupled system admits viscosity subsolutions.

We then study the corresponding *critical weakly coupled system* and show that the obstruction to the existence of a globally strict subsolution is not spread indistinctly on the torus, but concentrates on a closed set $\mathcal{A}$, that we call *Aubry set* in analogy to the case of a single critical equation. In particular, we show the existence of a *critical subsolution* (i.e. a viscosity subsolution of the critical weakly coupled system), which is strict outside the Aubry set. Furthermore, any such subsolution can be taken of class C^∞ in the complementary of $\mathcal{A}$. This is achieved by exploiting the regularization procedure presented in [19, 20] in the case of a single equation. As a byproduct, we obtain that all such subsolutions form a dense subset of the family of critical subsolutions.

The analysis outlined above allows us to derive in a neat way a comparison result among critical sub and supersolutions satisfying suitable "boundary" conditions on $\mathcal{A}$, see Theorem 5.3. This generalizes to our setting Theorem 3.3 in [5], therein established for Hamiltonians of Eikonal type. In particular, we derive that the Aubry set is a uniqueness set for the critical weakly coupled system. We furthermore show that the trace of any critical subsolution on $\mathcal{A}$ can be extended on the whole torus in such a way that the output is a critical solution, see Theorem 5.5.

Our study highlights a rigidity phenomenon taking place on the Aubry set, see Theorem 5.1: any two critical subsolutions differ, at each point y of $\mathcal{A}$, by a vector of the form $k\,(1, 1, \dots, 1)$, where k is a real number depending on y and on the critical subsolutions itself. This means that the differences of the components of a critical subsolution on $\mathcal{A}$ are actually independent of the specific subsolution chosen. The proof relies on Proposition 2.3, which is a key remark for this and other results in

the paper and which depends in a crucial way on the irreducible hypothesis assumed on the matrix $B(x)$. In the particular case when there exists a critical subsolution of the kind $\big(v(x), v(x), \ldots, v(x)\big)$, we infer that any other critical subsolution is of this form on $\mathcal{A}$. This accounts for the kind of symmetry already observed, in a weaker form, in [5] for the particular class of Hamiltonians therein considered, see Section 6.1 for more details.

A second rigidity phenomenon that we point out is when the Hamiltonians are additionally assumed strictly convex in the momentum: in this case we prove that, at any point of the Aubry set, the intersection of the reachable gradients of all the critical subsolutions is always nonempty, see Proposition 4.4. This can be regarded as a weak version, for weakly coupled systems, of a result holding in the case of a single equation. Under suitable regularity assumptions on the Hamiltonian, it is in fact well known that the subsolutions of the critical equation (2) are all differentiable on the Aubry set and have the same gradient, see [18, 19, 20].

We end our study by presenting a couple of situations where more explicit results can be obtained for the critical value and for the Aubry set, see Section 6. In the first example, we focus on the setting studied in [5] and we show that our notion of Aubry set is consistent with the one therein given. The second example contains, as a particular instance, the case when the Hamiltonians are all equal, say to H. In this specific situation, we show that the critical value and the Aubry set of the weakly coupled system agree with the critical value and the Aubry set of H. We furthermore show that the solutions of the critical system are all of the form $u(x)(1, \ldots, 1)$, where u is a critical solution for H, thus showing that, as far as critical solutions of the system are concerned, the coupling is not playing any effective role.

Weakly coupled systems are a particular instance of monotone systems, see [4, 13, 23, 24], but our interest for the issues herein considered stems from a series of recent papers [5, 25, 26, 28] addressed to understand the long–time behavior of the solutions $\big(u_1(t, x), \ldots, u_m(t, x)\big)$ to the evolutionary counterpart of (3), i.e. the system

$$\frac{\partial u_i}{\partial t} + H_i(x, D_x u_i) + \sum_{j=1}^{m} b_{ij}(x) u_j(t, x) = 0 \quad \text{in } (0, +\infty) \times \mathbb{T}^N, \quad \forall i \in \{1, \ldots, m\}. \tag{4}$$

The issue is to find general conditions on the Hamiltonians and on the coupling matrix sufficient to guarantee that

$$u_i(t, x) + ct \underset{t \to +\infty}{\rightrightarrows} v_i \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \ldots, m\},$$

where c is the critical value and $\big(v_1(x), \ldots, v_m(x)\big)$ is a critical solution. The problem is non–trivial due to the lack of a strong comparison principle for the critical system.

The case of a single equation, i.e. equation (4), is by now well understood. The first convergence results were obtained by Namah and Roquejoffre [27] for convex Hamiltonians of Eikonal type, but the breakthrough is due to Fathi [17], maybe historically one of the first PDE achievements of weak KAM theory. By making use of the dynamical insight, Fathi established the long–time convergence for Hamiltonians of Tonelli type, i.e. smooth, strictly convex and superlinear in the momentum variable. Related results can be also found in [29]. The dynamical approach was subsequently relaxed to much less regular Hamiltonians (continuous, coercive and strictly convex in the momentum variable) by Davini and Siconolfi [10], then simplified and extended by Ishii [22] to the non–compact setting. A completely different

approach, based on PDE methods, was instead proposed by Barles and Souganidis [2]: they establish the long–time convergence for continuous Hamiltonians that are coercive and satisfy a convex–type inequality with respect to the momentum variable. This condition includes the case of Tonelli Hamiltonians and also some nonconvex functions.

Thanks to the qualitative analysis available for the critical equation, the asymptotic problem can be reduced to studying the convergence on the Aubry set. The issue is however subtle: the afore mentioned references reveal that, in order to have a general convergence result, some kind of strict convexity in the momentum has to be required on the Hamiltonian.

The question whether a similar statement holds for weakly coupled systems was first raised in [5] and answered positively by considering a setting analogous to [27]. More precisely, the Hamiltonians are assumed of the form

$$H_i(x,p) := F_i(x,p) - V_i(x) \qquad \text{for every } (x,p) \in \mathbb{T}^N \times \mathbb{R}^N \text{ and } i \in \{1,\dots,m\},$$

where each V_i is a continuous, non–negative function on $\mathbb{T}^N$ and each F_i is a continuous function on $\mathbb{T}^N \times \mathbb{R}^N$, convex and coercive in p, and satisfying

$$F_i(x,p) \geqslant F_i(x,0) = 0 \quad \text{for every } (x,p) \in \mathbb{T}^N \times \mathbb{R}^N.$$

The asymptotic convergence result is proved by assuming that the set

$$\mathcal{A}_0 := \bigcap_{i=1}^{m} V_i^{-1}(\{0\}) \cap \big\{ x \in \mathbb{T}^N \,:\, B(x) \text{ is degenerate} \big\}$$

is non–empty (note that here the coupling matrix is not assumed degenerate on the whole $\mathbb{T}^N$). In particular, this implies that the critical value c is equal to 0. The authors first prove that $\mathcal{A}_0$ is a uniqueness set for the critical weakly coupled system with respect to suitable "boundary" conditions on $\mathcal{A}_0$. Then they exploit the special structure of the Hamiltonians to show that an appropriate linear combination of the components of the time–dependent solution asymptotically converges on $\mathcal{A}_0$, and this is actually enough to conclude. Similar results have been also obtained in [26].

A significant step forward has been recently taken by Mitake and Tran [25] and by Nguyen [28], independently. In both papers, the coupling matrix is assumed independent of x and degenerate.

In [25], the convergence is established in the case $m = 2$ for continuous Hamiltonians that are strictly convex and coercive in the momentum variable. The analysis is based on an interesting dynamic programming formula for systems whose states are governed by random changes and on an adaptation of the techniques exploited in [10, 22].

The same result has been proved in [28] for systems with an arbitrary number of equations through a completely different approach based on PDE techniques. Furthermore, the author is able to extend the convergence result under a different set of assumptions in the same spirit of [2], thus including some examples of non–convex Hamiltonians.

The case of a coupling matrix depending on x is widely open. The analysis carried out herein sheds some light on the kind of phenomena occurring for the stationary system at the critical level and, we hope, might be useful for further investigations on the subject.

This paper is organized as follows. Section 1.1 contains the notation. In Section 1.2 we present some basic results of linear algebra on coupling matrices. In Section

1.3, we precise the setting and the assumptions adopted throughout the paper, and we present a brief overview of existing results on weakly coupled systems. In Section 2 we give the definition and establish some relevant properties of the critical value. In particular, we prove the existence of viscosity solutions to the critical weakly coupled system, see Theorem 2.12. Some proofs are postponed to the Appendix A. In Section 3 we give the definition of Aubry set and explore its properties. This is done by introducing an analogue of the Mañé potential, the Mañé matrix. The first part of Section 4 is devoted to the regularization of subsolutions outside of the Aubry set. In the second part, we prove the rigidity phenomenon enjoyed by the reachable gradients of the critical subsolutions previously described. The fact that the values assumed on the Aubry set by a critical subsolution depend very less on the subsolution itself is instead proved at the beginning of Section 5. In the remainder of the section we state and prove the comparison principle, Theorem 5.3, and we show how the trace of a critical subsolution on the Aubry set can be extended to the whole torus to produce a critical solution, see Theorem 5.5. Last, in Section 6 we illustrate our theory on two examples.

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1. Preliminaries

1.1. **Notations.** Throughout the paper, we will denote by $\mathbb{T}^N$ the N–dimensional flat torus, where N is an integer number. The scalar product in $\mathbb{R}^N$ will be denoted by $\langle \cdot\,,\cdot \rangle$, while the symbol $|\cdot|$ stands for the Euclidean norm. Note that the latter induces a norm on $\mathbb{T}^N$, still denoted by $|\cdot|$, defined as

$$|x| := \min_{\kappa\in\mathbb{Z}^N} |x+\kappa| \qquad \text{for every } x\in\mathbb{T}^N.$$

We will denote by $B_R(x_0)$ and B_R the closed balls in $\mathbb{T}^N$ of radius R centered at x_0 and 0, respectively.

With the symbol $\mathbb{R}_+$ we will refer to the set of nonnegative real numbers. We say that a property holds *almost everywhere* (*a.e.* for short) in a subset E of $\mathbb{T}^N$ if it holds up to a *negligible* subset of E, i.e. a subset of zero N–dimensional Lebesgue measure.

By modulus we mean a nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$, vanishing and continuous at 0. A function $g:\mathbb{R}_+\to\mathbb{R}$ will be termed *coercive* if $\lim\limits_{h\to+\infty} g(h)=+\infty$.

We will say that $(\rho_n)_n$ is a sequence of *standard mollifiers* if $\rho_n(x):=n^N\rho(nx)$ in $\mathbb{R}^N$ for each $n\in\mathbb{N}$, where ρ is a smooth, non–negative function on $\mathbb{R}^N$, supported in B_1 and such that its integral over $\mathbb{R}^N$ is equal to 1.

Given a continuous function u on $\mathbb{T}^N$, we will call *subtangent* (respectively, *supertangent*) of u at x_0 a function ϕ of class C^1 in a neighborhood U of x_0 such that $\phi(x_0)=u(x_0)$ and $\phi(x)\leqslant u(x)$ for every $x\in U$ (resp., $\geqslant$). Its gradient $D\phi(x_0)$ will be called a *subdifferential* (resp. *superdifferential*) of u at x_0, respectively. The set of sub and superdifferentials of u at x_0 will be denoted $D^-u(x_0)$ and $D^+u(x_0)$,

respectively. We recall that u is differentiable at x_0 if and only if $D^+u(x_0)$ and $D^-u(x_0)$ are both nonempty. In this instance, $D^+u(x_0) = D^-u(x_0) = \{Du(x_0)\}$, where $Du(x_0)$ denotes the differential of u at x_0. We refer the reader to [6] for the proofs.

When u is locally Lipschitz in $\mathbb{T}^N$, we will denote by $\partial^* u(x_0)$ the set of *reachable gradients* of u at x_0, that is the set

$$\partial^* u(x_0) = \{\lim_n Du(x_n) \,:\, u \text{ is differentiable at } x_n,\, x_n \to x_0\,\},$$

while the *Clarke's generalized gradient* $\partial_c u(x_0)$ is the closed convex hull of $\partial^* u(x_0)$. The set $\partial_c u(x_0)$ contains both $D^+u(x_0)$ and $D^-u(x_0)$, in particular $Du(x_0) \in \partial_c u(x_0)$ at any differentiability point x_0 of u. We refer the reader to [8] for a detailed treatment of the subject.

We will denote by $\|g\|_\infty$ the usual L^∞–norm of g, where the latter is a measurable real function defined on $\mathbb{T}^N$. We will write $g_n \rightrightarrows g$ in $\mathbb{T}^N$ to mean that the sequence of functions $(g_n)_n$ uniformly converges to g in $\mathbb{T}^N$. We will denote by $\big(\mathrm{C}(\mathbb{T}^N)\big)^m$ the Banach space of continuous functions $\mathbf{u} = (u_1, \ldots, u_m)^T$ from $\mathbb{T}^N$ to $\mathbb{R}^m$, endowed with the norm

$$\|\mathbf{u} - \mathbf{v}\|_\infty = \max_{1 \leqslant i \leqslant m} \|u_i - v_i\|_\infty, \qquad \mathbf{u}, \mathbf{v} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m.$$

We will write $\mathbf{u}^n \rightrightarrows \mathbf{u}$ in $\mathbb{T}^N$ to mean that $\|\mathbf{u}^n - \mathbf{u}\|_\infty \to 0$. A function $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ will be termed Lipschitz continuous if each of its components is κ–Lipschitz continuous, for some $\kappa > 0$. Such a constant κ will be called a *Lipschitz constant* for $\mathbf{u}$. The space of all such functions will be denoted by $\big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$.

We will denote by $\mathbb{1} = (1, \cdots, 1)^T$ the vector of $\mathbb{R}^m$ having all components equal to 1, where the upper–script symbol T stands for the transpose. We consider the following partial relations between elements $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$: $\mathbf{a} \leqslant \mathbf{b}$ if $a_i \leqslant b_i$ (resp., $<$) for every $i \in \{1, \ldots, m\}$. Given two functions $\mathbf{u}, \mathbf{v} : \mathbb{T}^N \to \mathbb{R}^m$, we will write $\mathbf{u} \leqslant \mathbf{v}$ in $\mathbb{T}^N$ (respectively, $<$) to mean that $\mathbf{u}(x) \leqslant \mathbf{v}(x)$ $\big($resp., $\mathbf{u}(x) < \mathbf{v}(x)\big)$ for every $x \in \mathbb{T}^N$.

1.2. **Linear algebra.** Here we briefly present some elementary linear algebraic results concerning coupling matrices.

Definition 1.1. *Let $B = (b_{ij})_{i,j}$ be a $m \times m$–matrix.*

(i) We say that B is a coupling matrix *if it satisfies the following conditions:*

$$b_{ij} \leqslant 0 \text{ for } j \neq i, \qquad \sum_{j=1}^m b_{ij} \geqslant 0 \qquad \text{for any } i \in \{1, \ldots, m\}. \tag{C}$$

It is additionally termed degenerate *if*

$$\sum_{j=1}^m b_{ij} = 0 \qquad \text{for any } i = 1, \ldots, m.$$

(ii) We say that B is irreducible *if for every subset $\mathcal{I} \subsetneq \{1, \ldots, m\}$ there exist $i \in \mathcal{I}$ and $j \notin \mathcal{I}$ such that $b_{ij} \neq 0$.*

When a coupling matrix is also irreducible, further information can be derived on its elements. We have

Proposition 1.2. *Let $B = (b_{ij})_{i,j}$ be an irreducible $m \times m$ coupling matrix. Then* $b_{ii} > 0$ *for every $i \in \{1, \ldots, m\}$.*

Proof. Indeed, if $b_{i_0 i_0} = 0$ for some $i_0 \in \{1, \ldots, m\}$, condition (C) would imply $b_{i_0 j} = 0$ for every $j \in \{1, \ldots, m\}$, in contradiction with the fact that B is irreducible. □

The following invertibility criterion holds:

Proposition 1.3. *Let $B = (b_{ij})_{i,j}$ be an $m \times m$ irreducible coupling matrix. Then*

(i) $\mathrm{Ker}(B) \subseteq \mathrm{span}\{(1, \ldots, 1)^T\} = \mathbb{R}\mathbb{1}$;

(ii) $\mathrm{Ker}(B) = \mathrm{span}\{(1, \ldots, 1)^T\} = \mathbb{R}\mathbb{1}$ *if and only if* B *is degenerate.*

In particular, B is invertible if and only if

$$\sum_{j=1}^{m} b_{ij} > 0 \qquad \textit{for some } i \in \{1, \ldots, m\}.$$

Proof. We first remark that, by assumption (C),

$$b_{ii} \geqslant \sum_{j \neq i} |b_{ij}| \qquad \text{for every } i \in \{1, \ldots, m\}. \tag{5}$$

Let us prove *(i)*. Let $\mathbf{v} = (v_1, \ldots, v_m)^T \in \mathrm{Ker}(B)$ and set

$$\mathcal{I} = \big\{ i \in \{1, \ldots, m\} \,:\, v_i = \max\{v_1, \ldots, v_m\} \big\}.$$

We claim that $\mathcal{I} = \{1, \ldots, m\}$. Indeed, if this were not the case, by the irreducible character of B there would exist $i \in \mathcal{I}$ and $k \notin \mathcal{I}$ such that $b_{ik} \neq 0$. Since $B\mathbf{v} = 0$, we would get in particular

$$b_{ii} v_i = \sum_{j \neq i} v_j |b_{ij}| \leqslant v_i \sum_{j \neq i} |b_{ij}| \leqslant v_i b_{ii}.$$

Then the inequalities must be equalities. We infer

$$v_j |b_{ij}| = v_i |b_{ij}| \qquad \text{for every } j \neq i,$$

in particular $v_k = v_i = \max\{v_1, \ldots, v_m\}$, yielding that k belongs to $\mathcal{I}$, a contradiction.

The remainder of the statement trivially follows from item *(i)*. □

The following proposition gives an obstruction to being in the image of a degenerate coupling matrix.

Proposition 1.4. *Let B a coupling and degenerate $m \times m$ matrix. If $\mathbf{a} = (a_1, \ldots, a_m)^T$ satisfies $a_i > 0$ for every $i \in \{1, \ldots, m\}$, then $\mathbf{a} \notin \mathrm{Im}(B)$.*

Proof. Let us assume by contradiction that there exists $\mathbf{v} = (v_1, \ldots, v_m)^T$ such that

$$B\mathbf{v} = \mathbf{a}.$$

Let $v_k = \min\{v_1, \ldots, v_m\}$. We have

$$a_k = \sum_{j=1}^{m} b_{kj} v_j \leqslant \sum_{j=1}^{m} b_{kj} v_k = 0,$$

in contradiction with the hypothesis $a_k > 0$. □

1.3. **Weakly coupled systems.** Throughout the paper, we will call *convex Hamiltonian* a function H satisfying the following set of assuptions:

(H1) $H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous;

(H2) $p \mapsto H(x,p)$ is convex on $\mathbb{R}^N$ for any $x \in \mathbb{T}^N$;

(H3) there exist two coercive functions $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$ such that
$$\alpha(|p|) \leqslant H(x,p) \leqslant \beta(|p|) \qquad \text{for all } (x,p) \in \mathbb{T}^N \times \mathbb{R}^N.$$

The Hamiltonian H will be termed *strictly convex* if it additionally satisfies the following stronger assumption:

(H2)′ $p \mapsto H(x,p)$ is strictly convex on $\mathbb{R}^N$ for any $x \in \mathbb{T}^N$.

Moreover, we will denote by $B(x) = \big(b_{ij}(x)\big)_{i,j}$ an $m \times m$–matrix with continuous coefficients $b_{ij}(x)$ on $\mathbb{T}^N$. If not otherwise stated, the following hypotheses will be always assumed:

(B1) $B(x)$ is an irreducible coupling matrix for every $x \in \mathbb{T}^N$;

(B2) $B(x)$ is degenerate for every $x \in \mathbb{T}^N$.

Let $H_1(x,p), \dots, H_m(x,p)$ be convex Hamiltonians, i.e. functions satisfying conditions (H1)–(H3). We are interested in weakly coupled systems of the form
$$H_i(x, Du_i) + \big(B(x)\mathbf{u}(x)\big)_i = a_i \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \dots, m\}, \tag{6}$$
for some constant vector $\mathbf{a} = (a_1, \dots, a_m)^T$, where $\mathbf{u}(x) = \big(u_1(x), \dots, u_m(x)\big)^T$ and $\big(B(x)\mathbf{u}(x)\big)_i$ denotes the i–th component of the vector $B(x)\mathbf{u}(x)$, i.e.
$$\big(B(x)\mathbf{u}(x)\big)_i = \sum_{j=1}^{m} b_{ij}(x) u_j(x).$$

Remark 1.5. The weakly coupled system (6) is a particular type of *monotone system*, i.e. a system of the form
$$G_i\big(x, u_1(x), \dots, u_m(x), Du_i\big) = 0 \qquad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \dots, m\},$$
where suitable monotonicity conditions with respect to the u_j–variables are assumed on the functions G_i, see [4, 13, 21, 23, 24]. In the specific case considered in this paper, the conditions assumed on the coupling matrix imply, in particular, that each function G_i is strictly increasing in u_i and non–increasing in u_j for every $j \neq i$. This kind of monotonicity will be exploited in many points of the paper.

Let $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$. We will say that $\mathbf{u}$ is a *viscosity subsolution* of (6) if the following inequality holds for every $(x,i) \in \mathbb{T}^N \times \{1, \dots, m\}$
$$H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a_i \quad \text{for every } p \in D^+u_i(x).$$
We will say that $\mathbf{u}$ is a *viscosity supersolution* of (6) if the following inequality holds for every $(x,i) \in \mathbb{T}^N \times \{1, \dots, m\}$
$$H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \geqslant a_i \quad \text{for every } p \in D^-u_i(x).$$
We will say that $\mathbf{u}$ is a *viscosity solution* if it is both a sub and a supersolution. In the sequel, solutions, subsolutions and supersolutions will be always meant in the viscosity sense, hence the adjective *viscosity* will be omitted.

Due to the convexity of the Hamitonian H_i, the following equivalences hold:

Proposition 1.6. *Let $a \in \mathbb{R}$, $i \in \{1, \dots, m\}$ and $\mathbf{u} \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$. The following facts are equivalent:*

(i) $H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a$ *for every $p \in D^+ u_i(x)$ and $x \in \mathbb{T}^N$;*

(ii) $H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a$ *for every $p \in D^- u_i(x)$ and $x \in \mathbb{T}^N$;*

(iii) $H_i(x,p) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a$ *for every $p \in \partial_c u_i(x)$ and $x \in \mathbb{T}^N$;*

(iv) $H_i\big(x, Du_i(x)\big) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a$ *for a.e. $x \in \mathbb{T}^N$.*

Next, we state a proposition that will be needed in the sequel, see also [13, 24, 21, 23] for similar results.

Proposition 1.7. *Let $\mathcal{F}$ be a subset of $\big(\mathrm{C}(\mathbb{T}^N)\big)^m$ and define the functions $\underline{\mathbf{u}}$, $\overline{\mathbf{u}}$ on $\mathbb{T}^N$ by setting:*

$$\underline{u}_i(x) = \inf_{\mathbf{u} \in \mathcal{F}} u_i(x), \quad \overline{u}_i(x) = \sup_{\mathbf{u} \in \mathcal{F}} u_i(x) \qquad \text{for every } x \in \mathbb{T}^N \text{ and } i \in \{1, \dots, m\}.$$

Assume that $\underline{\mathbf{u}}$ and $\overline{\mathbf{u}}$ belong to $\big(\mathrm{C}(\mathbb{T}^N)\big)^m$ and let $\mathbf{a} \in \mathbb{R}^m$. Then:

(i) if every $\mathbf{u} \in \mathcal{F}$ is a subsolution of (6), then $\overline{\mathbf{u}}$ is a subsolution of (6);

(ii) if every $\mathbf{u} \in \mathcal{F}$ is a supersolution of (6), then $\underline{\mathbf{u}}$ is a supersolution of (6).

The previous proposition is analogous to a well known fact for scalar Hamilton–Jacobi equations, see for instance Section 2.6 in [1]. The proof can be easily recovered by arguing similarly and by exploiting the monotonicity of the system.

We will be also interested in the evolutionary counterpart of (6), i.e. the system

$$\frac{\partial u_i}{\partial t} + H_i(x, D_x u_i) + \big(B(x)\mathbf{u}(t,x)\big)_i = 0 \quad \text{in } (0, +\infty) \times \mathbb{T}^N \qquad \forall i \in \{1, \dots, m\}, \tag{7}$$

where we have denoted by $\mathbf{u}(t,x) = \big(u_1(t,x), \dots, u_m(t,x)\big)^T$.

The following comparison result holds, see for instance [4] for a proof.

Proposition 1.8. *Let $T > 0$ and $\mathbf{v}$, $\mathbf{u} \in \big(\mathrm{Lip}([0,T] \times \mathbb{T}^N)\big)^m$ be, respectively, a sub and a supersolution of (7). Then, for every $i \in \{1, \dots, m\}$,*

$$v_i(t,x) - u_i(t,x) \leqslant \max_{1 \leqslant i \leqslant m} \max_{\mathbb{T}^N} \big(v_i(0,\cdot) - u_i(0,\cdot)\big), \qquad (t,x) \in [0,T] \times \mathbb{T}^N.$$

By making use of this proposition and of Perron's method, it is then easy to prove the following

Proposition 1.9. *Let $\mathbf{u}_0 \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$. Then there exists a unique function $\mathbf{u}(t,x)$ in $\big(\mathrm{Lip}(\mathbb{R}_+ \times \mathbb{T}^N)\big)^m$ that solves the system (7) subject to the initial condition $\mathbf{u}(0,x) = \mathbf{u}_0(x)$ in $\mathbb{T}^N$. Moreover, the Lipschitz constant of $\mathbf{u}(t,x)$ in $\mathbb{R}_+ \times \mathbb{T}^N$ only depends on the Hamiltonians $H_1, \dots, H_m$ and on the Lipschitz constant of $\mathbf{u}_0$.*

We will denote by $\mathcal{S}(t)\mathbf{u}_0(x)$ the solution $\mathbf{u}(t,x)$ of (7) with initial datum $\mathbf{u}_0$. This defines, for every $t > 0$, a map

$$\mathcal{S}(t) : \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m \to \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m.$$

We summarize in the next proposition the properties enjoyed by such maps, which come as an easy application of the above results.

Proposition 1.10. *For every $t, s > 0$ and $\mathbf{u}, \mathbf{v} \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$ we have:*

(i) **(Semigroup property)** $\mathcal{S}(s)\big(\mathcal{S}(t)\mathbf{u}\big) = \mathcal{S}(t+s)\mathbf{u}$ *in* $\mathbb{T}^N$*;*

(ii) **(Monotonicity)** *if* $\mathbf{v} \leqslant \mathbf{u}$ *in* $\mathbb{T}^N$*, then* $\mathcal{S}(t)\mathbf{v} \leqslant \mathcal{S}(t)\mathbf{u}$ *in* $\mathbb{T}^N$*;*

(iii) **(Non–expansiveness property)** $\|\mathcal{S}(t)\mathbf{v} - \mathcal{S}(t)\mathbf{u}\|_\infty \leqslant \|\mathbf{v} - \mathbf{u}\|_\infty$*;*

(iv) for every $a \in \mathbb{R}$, $\mathcal{S}(t)(\mathbf{u} + a\mathbb{1}) = \mathcal{S}(t)\mathbf{u} + a\mathbb{1}$ *in* $\mathbb{T}^N$*.*

The fact that the coupling matrix $B(x)$ is everywhere degenerate is crucial for assertion *(iv)*.

2. The critical value

The purpose of this section is to define the notion of critical value for weakly coupled systems and to prove some relevant properties of the corresponding critical system.

We start by proving some *a priori* estimates for the subsolutions of a weakly coupled system of the form (6). The following notation will be assumed throughout the section:

$$\mu_i = \min_{(x,p)} H_i(x,p) \quad \text{for each } i \in \{1, \dots, m\}, \qquad \mu = \min_{i \in \{1,\dots,m\}} \mu_i.$$

Proposition 2.1. *Let $\mathbf{a} = (a_1, \dots, a_m)^T \in \mathbb{R}^m$ and $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ such that*

$$\big(B(x)\mathbf{u}(x)\big)_i \leqslant a_i \qquad \text{for every } x \in \mathbb{T}^N \text{ and } i \in \{1, \dots, m\}. \tag{8}$$

Then there exists a constant $M_{\mathbf{a}}$ only depending on $\mathbf{a}$ and $B(x)$ such that

(i) $\|u_i - u_j\|_\infty \leqslant M_{\mathbf{a}}$ *for every i, $j \in \{1, \dots, m\}$;*

(ii) $\big|\big(B(x)\mathbf{u}(x)\big)_i\big| \leqslant M_{\mathbf{a}}$ *for every $x \in \mathbb{T}^N$ and $i \in \{1, \dots, m\}$.*

Proof. It suffices to prove the assertion for $\mathbf{a} = a\,\mathbb{1}$. Let us set

$$\beta_\star = \min_{1 \leqslant i \leqslant m} \min_{x \in \mathbb{T}^N} b_{ii}(x), \qquad \beta^\star = \max_{1 \leqslant i,j \leqslant m} \max_{x \in \mathbb{T}^N} |b_{ij}(x)|.$$

Such quantities are finite valued. Moreover, $\beta_\star$ is strictly positive in view of Proposition 1.2 and of the fact that $B(x)$ is, for every $x \in \mathbb{T}^N$, an irreducible coupling matrix with continuous coefficients.

Let us now fix $x \in \mathbb{T}^N$ and assume, without any loss of generality,

$$u_1(x) \leqslant u_2(x) \leqslant \cdots \leqslant u_m(x). \tag{9}$$

First notice that, by subtracting $\sum_{j=1}^m b_{mj}(x)u_m(x) = 0$ from both sides of equation (8) with $i = m$, one gets

$$\sum_{j \neq m} -b_{mj}(x)\big(u_m(x) - u_j(x)\big) \leqslant a,$$

yielding

$$\Big(u_m(x) - \max_{j \neq m} u_j(x)\Big) \sum_{j \neq m} -b_{mj}(x) \leqslant a.$$

By exploiting (9) and the degenerate character of the matrix $B(x)$ we get

$$0 \leqslant u_m(x) - u_{m-1}(x) \leqslant \frac{a}{b_{mm}(x)} \leqslant \frac{a}{\beta_\star}. \tag{10}$$

This proves assertion *(i)* when $m=2$. To prove it in the general case, we argue by induction: we assume the result true for m and we prove it for $m+1$. To this aim, we restate equation (8) as

$$\sum_{j=1}^{m-1} b_{ij}(x)u_j(x) + \Big(b_{im}(x)+b_{i\,m+1}(x)\Big)u_m(x) + b_{i\,m+1}(x)\Big(u_{m+1}(x)-u_m(x)\Big) \leqslant a,$$

then we exploit (10) to get

$$\sum_{j=1}^{m-1} b_{ij}(x)u_j(x) + \Big(b_{im}(x)+b_{i\,m+1}(x)\Big)u_m(x) \leqslant a\left(1+\frac{\beta^\star}{\beta_\star}\right) \tag{11}$$

for every $i \in \{1,\dots,m+1\}$. The irreducible character of $B(x)$ applied to the set $\mathcal{I}=\{m,m+1\}$ implies that

$$b_{im}(x)+b_{i\,m+1}(x)>0$$

for either $i=m$ or $i=m+1$, let us say $i=m$ for definitiveness. Assertion *(i)* now follows by applying the induction hypothesis to the system given by (11) with i varying in $\{1,\dots,m\}$, the corresponding coupling matrix being still irreducible and degenerate.

To prove *(ii)* it suffices to note that, for every $i \in \{1,\dots,m\}$,

$$\begin{aligned} -\big(B(x)\mathbf{u}(x)\big)_i &= -b_{ii}(x)u_i(x) + \sum_{j\neq i}\big(-b_{ij}(x)\big)u_j(x) \\ &\leqslant -b_{ii}(x)u_i(x) + \sum_{j\neq i} -b_{ij}(x)\big(u_i(x)+\|u_i-u_j\|_\infty\big) \\ &\leqslant (m-1)\,\beta^\star\|u_i-u_j\|_\infty, \end{aligned}$$

and the assertion follows from *(i)* and from hypothesis (8). □

As a consequence, we derive the following result:

Proposition 2.2. *Let $\mathbf{u}=(u_1,\dots,u_m)^T \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ be a subsolution of* (6) *for some $\mathbf{a}\in\mathbb{R}^m$. Then there exists costants $C_{\mathbf{a}}$ and $\kappa_{\mathbf{a}}$, only depending on $\mathbf{a}$, H_i and the coupling matrix $B(x)$, such that*

(i) $\|u_i-u_j\|_\infty \leqslant C_{\mathbf{a}}$ *for every i, $j\in\{1,\dots,m\}$;*

(ii) $\mathbf{u}$ *is $\kappa_{\mathbf{a}}$–Lipschitz continuous in $\mathbb{T}^N$.*

Proof. For each $i\in\{1,\dots,m\}$, the following inequalities hold in the viscosity sense:

$$\mu + \big(B(x)\mathbf{u}(x)\big)_i \leqslant H_i(x,Du_i) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a_i \qquad \text{in } \mathbb{T}^N,$$

yielding

$$\big(B(x)\mathbf{u}(x)\big)_i \leqslant a_i-\mu \qquad \text{for every } x\in\mathbb{T}^N.$$

In view of Proposition 2.1 we get *(i)* and

$$\Big|\big(B(x)\mathbf{u}(x)\big)_i\Big| \leqslant M_{\mathbf{a}} \qquad \text{for every } x\in\mathbb{T}^N.$$

Plugging this inequality in (6) we derive that u_i is a viscosity subsolution of

$$H_i(x,Du_i)\leqslant a_i+M_{\mathbf{a}} \qquad \text{in } \mathbb{T}^N$$

and assertion *(ii)* follows as well via a standard argument that exploits the coercivity of $H_i(x,p)$ in p. □

Next, we establish a remarkable property of weakly coupled systems.

Proposition 2.3. *Assume that* $\mathbf{v}$, $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ *are, respectively a sub and a supersolution of the weakly coupled system* (6) *for some* $\mathbf{a} \in \mathbb{R}^m$. *Let* $x_0 \in \mathbb{T}^N$ *be such that*

$$v_i(x_0) - u_i(x_0) = M := \max_{1 \leqslant i \leqslant m} \max_{\mathbb{T}^N} (v_i - u_i) \qquad \textit{for some } i \in \{1, \dots, m\}.$$

Then $\mathbf{v}(x_0) = \mathbf{u}(x_0) + M\mathbb{1}$.

Proof. In view of Proposition 2.2, we know that $\mathbf{v}$ is Lipschitz continuous. Set

$$\mathcal{I} = \big\{ i \in \{1, \dots, m\} \, : \, \big(v_i(x_0) - u_i(x_0)\big) = M \, \big\}.$$

We want to prove that $\mathcal{I} = \{1, \dots, m\}$. Indeed, if this were not the case, by the irreducible character of the matrix $B(x_0)$ there would exist $i \in \mathcal{I}$ and $k \notin \mathcal{I}$ such that

$$b_{ik}(x_0) < 0.$$

We now make use of the method of doubling the variables to reach a contradiction. For every $\varepsilon > 0$, we set

$$\psi^\varepsilon(x, y) = v_i(x) - u_i(y) - \frac{|x - y|^2}{2\varepsilon^2} - \frac{|x - x_0|^2}{2}, \qquad x, y \in \mathbb{T}^N.$$

Let $M_\varepsilon = \max_{\mathbb{T}^N \times \mathbb{T}^N} \psi_\varepsilon$ and denote by $(x_\varepsilon, y_\varepsilon)$ a point in $\mathbb{T}^N \times \mathbb{T}^N$ where such a maximum is achieved. By a standard argument in the theory of viscosity solution, see for instance Lemma 2.3 in [1], the following properties hold:

$$x_\varepsilon, y_\varepsilon \to x_0, \quad \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} \to 0, \quad |x_\varepsilon - x_0| \to 0 \qquad \text{as } \varepsilon \to 0. \tag{12}$$

Furthermore,

$$p'_\varepsilon := \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} \in D^- u_i(y_\varepsilon), \quad p_\varepsilon := p'_\varepsilon - (x_\varepsilon - x_0) \in D^+ v_i(x_\varepsilon) \qquad \text{for every } \varepsilon > 0.$$

By the Lipschitz character of v_i we derive that the vectors $\{\, p_\varepsilon \, : \, \varepsilon > 0 \,\}$ are equi–bounded, hence, up to subsequences and in view of the estimates (12), we infer

$$p_\varepsilon, \, p'_\varepsilon \to p_0 \qquad \text{as } \varepsilon \to 0$$

for some vector $p_0 \in \mathbb{R}^N$. We now use the fact that $\mathbf{v}$ and $\mathbf{u}$ are a sub and supersolution of (16), respectively, to get

$$\begin{aligned} H_i(x_\varepsilon, p_\varepsilon) + \big(B(x_\varepsilon)\mathbf{v}(x_\varepsilon)\big)_i &\leqslant 0, \\ H_i(y_\varepsilon, p'_\varepsilon) + \big(B(y_\varepsilon)\mathbf{u}(y_\varepsilon)\big)_i &\geqslant 0. \end{aligned}$$

By subtracting the above inequalities and by passing to the limit for $\varepsilon \to 0$ we end up with

$$\Big(B(x_0)\big(\mathbf{v}(x_0) - \mathbf{u}(x_0)\big)\Big)_i \leqslant 0,$$

that is, since $i \in \mathcal{I}$ and the matrix $B(x_0)$ is degenerate,

$$M\, b_{ii}(x_0) \leqslant \sum_{j \neq i} |b_{ij}(x_0)| \big(v_j(x_0) - u_j(x_0)\big) \leqslant M \sum_{j \neq i} |b_{ij}(x_0)| = M b_{ii}(x_0).$$

Hence the above inequalities are equalities, in particular $v_k(x_0) - u_k(x_0) = M$ since $b_{ik}(x_0) \neq 0$, in contrast with the fact that $k \notin \mathcal{I}$. $\square$

Definition 2.4. For every $\mathbf{a} \in \mathbb{R}^m$, we denote by $\mathcal{H}(\mathbf{a})$ the set of subsolutions of the weakly coupled system (6). We will more simply write $\mathcal{H}(a)$ whenever $\mathbf{a} = a\mathbb{1}$ for some constant $a \in \mathbb{R}$, .

Lemma 2.5. *The sets $\mathcal{H}(\mathbf{a})$ are convex, closed (with respect to the topology of uniform convergence) and increasing with respect to the partial ordering on $\mathbb{R}^m$.*

Proof. Convexity and monotonicity are straightforward. The fact that the $\mathcal{H}(\mathbf{a})$ are closed is a direct consequence of stability of viscosity subsolutions. □

We now focus our attention to the case $\mathbf{a} = a\mathbb{1}$. As a direct consequence of the definition of the semigroup $\mathcal{S}(t)$, we get the following assertion:

Proposition 2.6. *Let $a \in \mathbb{R}$ and $\mathbf{u} \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)$. Then $\mathbf{u}$ is a viscosity solution of (6) with $\mathbf{a} = a\,\mathbb{1}$ if and only if*

$$\mathbf{u} = \mathcal{S}(t)\mathbf{u} + t\,a\mathbb{1} \quad \text{in } \mathbb{T}^N \qquad \text{for every } t > 0.$$

We have the following characterization:

Proposition 2.7. *Let $a \in \mathbb{R}$. The following facts are equivalent:*

(i) $\mathbf{u} \in \mathcal{H}(a)$;

(ii) *the map $t \mapsto \mathcal{S}(t)\mathbf{u} + t\,a\mathbb{1}$ is non–decreasing on $[0, +\infty)$.*

In particular, the sets $\mathcal{H}(a)$ are stable under the action of the semigroup $\mathcal{S}(t)$, in the sense that $\mathcal{S}(t)\big(\mathcal{H}(a)\big) \subset \mathcal{H}(a)$.

The proof of this proposition is rather technical and it is postponed to the Appendix A.

Definition 2.8. *The* critical value c *of the weakly coupled system* (6) *is defined as*

$$c = \inf\{a \in \mathbb{R} \,:\, \mathcal{H}(a) \neq \varnothing\}. \tag{13}$$

The following holds:

Proposition 2.9. *The critical value c is finite and $\mathcal{H}(c) \neq \varnothing$.*

Proof. By the growth assumptions on the Hamiltonians H_i it is easily seen that the function $\mathbf{u} \equiv (0, \dots, 0)^T$ is a subsolution of (6) for $a_0\mathbb{1}$ with $a_0 \in \mathbb{R}$ big enough.

Let us proceed to show that c is finite valued and that $\mathcal{H}(c) \neq \varnothing$. Let $(a_n)_n$ be a decreasing sequence converging to c and let $\mathbf{u}_n \in \mathcal{H}(c_n)$ for each $n \in \mathbb{N}$. Up to neglecting the first terms, we can assume that $a_n \leqslant a_0$ for every $n \in \mathbb{N}$. Arguing as in the proof of Proposition 2.2, we obtain that the following inequalities are satisfied in the viscosity sense:

$$\mu_i \leqslant H_i(x, Du_i^n) \leqslant a_n + M_{a_0} \qquad \text{in } \mathbb{T}^N$$

for every $i \in \{1, \dots, m\}$ and $n \in \mathbb{N}$, showing that c is finite. We now exploit Proposition 2.2: by the monotonicity of the sets $\mathcal{H}(a)$ with respect to a, we infer that the functions $\mathbf{u}_n$ are equi–Lipschitz. Up to subtracting a vector of the form $k_n\mathbb{1}$ to each $\mathbf{u}_n$, we can furthermore assume that $u_1^n(0) = 0$ for every $n \in \mathbb{N}$, yielding $\sup_n \|u_1^n\|_\infty \leqslant L$ for some $L \in \mathbb{R}$ by the equi–Lipschitz character of the sequence. Moreover,

$$\|u_j^n - u_1^n\|_\infty \leqslant C_{a_0} \qquad \text{for every } j \in \{1, \dots, m\} \text{ and } n \in \mathbb{N},$$

yielding

$$\|u_j^n\|_\infty \leqslant C_{a_0} + L \qquad \text{for every } j \in \{1, \dots, m\} \text{ and } n \in \mathbb{N}.$$

Up to subsequences, by the Arzela–Ascoli theorem, we infer that

$$\mathbf{u}^n \rightrightarrows \mathbf{u} \qquad \text{in } \mathbb{T}^N$$

and $\mathbf{u} \in \mathcal{H}(c)$ by stability of the notion of viscosity subsolution. $\square$

We now proceed to show that the stationary equation (3) possesses solutions of the corresponding weakly coupled system if and only if a equals the critical value c.

We start with a preliminary result.

Proposition 2.10. *Let $B(x)$ be a continuous irreducible coupling matrix on $\mathbb{T}^N$ and let us assume that $B(x)$ is invertible for every $x \in \mathbb{T}^N$. Let $\mathbf{v}$, $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ be, respectively, a sub and a supersolution of the weakly coupled system* (6)*, for some $\mathbf{a} \in \mathbb{R}^m$. Then*

$$\mathbf{v}(x) \leqslant \mathbf{u}(x) \qquad \textit{for every } x \in \mathbb{T}^N.$$

Proof. Arguing as in the proof of Proposition 2.2, we easily see that $\mathbf{v}$ is Lipschitz. Set

$$M = \max_{1 \leqslant i \leqslant m} \max_{\mathbb{T}^N} (v_i - u_i).$$

We want to prove that $M \leqslant 0$. Assume by contradiction that $M > 0$ and pick a point $x_0 \in \mathbb{T}^N$ where such a maximum is attained. Set

$$\mathcal{I} = \big\{ i \in \{1, \dots, m\} \,:\, \big(v_i(x_0) - u_i(x_0)\big) = M \,\big\}.$$

Arguing as in the proof of Proposition 2.3 we infer that

$$\Big(B(x_0)\big(\mathbf{v}(x_0) - \mathbf{u}(x_0)\big)\Big)_i \leqslant 0 \qquad \text{for every } i \in \mathcal{I}. \tag{14}$$

If $\mathcal{I} = \{1, \dots, m\}$, inequality (14) is in contradiction with the fact that $B(x_0)$ is invertible, in view of Proposition 1.3. If $\mathcal{I} \neq \{1, \dots, m\}$, we choose $i \in \mathcal{I}$ and $k \notin \mathcal{I}$ such that $b_{ik}(x_0) < 0$. From (14) and the assumption that $M > 0$ we infer that

$$M\, b_{ii}(x_0) \leqslant \sum_{j \neq i} |b_{ij}(x_0)| \big(v_j(x_0) - u_j(x_0)\big) \leqslant M \sum_{j \neq i} |b_{ij}(x_0)| \leqslant M b_{ii}(x_0),$$

which implies that $v_k(x_0) - u_k(x_0) = M$, in contrast with the fact that $k \notin \mathcal{I}$. $\square$

Next, we prove that solutions to a weakly coupled system of the kind (6) with $\mathbf{a} = a\mathbb{1}$ may exist only if a equals the critical value.

Proposition 2.11. *Let a, $b \in \mathbb{R}$ and $\mathbf{v}$, $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ such that the following inequalities are satisfied in the viscosity sense:*

$$\begin{aligned} H_i(x, Dv_i) + \big(B(x)\mathbf{v}(x)\big)_i &\leqslant a \qquad \textit{in } \mathbb{T}^N \\ H_i(x, Du_i) + \big(B(x)\mathbf{u}(x)\big)_i &\geqslant b \qquad \textit{in } \mathbb{T}^N \end{aligned}$$

for every $i \in \{1, \dots, m\}$. Then $b \leqslant a$.

Proof. Let us assume by contradiction that $b > a$. Up to replacing $\mathbf{v}$ with $\mathbf{v} + k\mathbb{1}$ with $k > 0$ big enough, we can assume

$$\mathbf{v} > \mathbf{u} \qquad \text{in } \mathbb{T}^N.$$

Let $\varepsilon > 0$ such that $b - \varepsilon > a + \varepsilon$. By continuity of the functions $\mathbf{v}$ and $\mathbf{u}$, we can find $\lambda > 0$ such that

$$\|\lambda\, v_i\|_\infty,\ \|\lambda\, u_i\|_\infty < \varepsilon \qquad \text{for every } i \in \{1, \dots, m\}.$$

Then the following inequalities hold in the viscosity sense in $\mathbb{T}^N$:

$H_i(x, Du_i) + \big((B(x) + \lambda\,\mathrm{I})\mathbf{u}(x)\big)_i > b - \varepsilon > a + \varepsilon > H_i(x, Dv_i) + \big((B(x) + \lambda\,\mathrm{I})\mathbf{v}(x)\big)_i.$

For ever $x \in \mathbb{T}^N$, the matrix $B(x) + \lambda\,\mathrm{I}$ is irreducible, satisfies (C) and the sum of the elements of each of its rows is strictly positive, hence it is invertible in view of Proposition 1.3. By Proposition 2.10 we conclude that

$$\mathbf{v} \leqslant \mathbf{u} \qquad \text{in } \mathbb{T}^N,$$

achieving a contradiction. □

We are now able to prove a weak KAM theorem, following the lines of Fathi [18]. This result has been already obtained in literature in similar settings by making use of the so called ergodic approximation, see [26, 5].

Theorem 2.12. *There exists a function $\mathbf{u} \in \mathcal{H}(c)$ that solves the weakly coupled system*

$$H_i(x, Du_i) + \big(B(x)\mathbf{u}(x)\big)_i = c \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \ldots, m\} \tag{15}$$

in the viscosity sense.

Proof. We have already proved in Proposition 2.9 that $\mathcal{H}(c) \neq \varnothing$. Let us introduce the quotient space $\hat{\mathcal{H}} = \mathcal{H}(c)\backslash\mathbb{R}\mathbb{1}$. Arguing as in the proof of Proposition 2.9, it is easily seen that $\hat{\mathcal{H}}$ is compact for the topology of uniform convergence. Indeed, it is isomorphic to the subset of $\mathcal{H}(c)$ of subsolutions whose first component vanishes at the point $x = 0$. Moreover, since the viscosity semigroup commutes with the addition of vectors of the form $\lambda\mathbb{1}$ and leaves $\mathcal{H}(c)$ stable, it induces a continuous semigroup, denoted $\hat{S}$, on $\hat{\mathcal{H}}$.

By the Schauder–Tychonoff fixed point theorem (see [12]), $\hat{S}$ possesses a fixed point, that is, there exists an element $\hat{\mathbf{u}} \in \hat{\mathcal{H}}$ such that

$$\forall t \geqslant 0, \quad \hat{S}(t)\hat{\mathbf{u}} = \hat{\mathbf{u}}.$$

Lifting these relations to $\mathcal{H}(c)$, we get

$$\forall t \geqslant 0 \text{ there exists } c_t \in \mathbb{R} \text{ such that } \mathcal{S}(t)\mathbf{u} = \mathbf{u} + c_t\mathbb{1},$$

where $\mathbf{u}$ is any element in the equivalence class of $\hat{\mathbf{u}}$. Since $\mathcal{S}$ is a semigroup, one readily realizes that the following relations are verified:

$$c_{t+s} = c_t + c_s \qquad \text{for every } t, s > 0.$$

Since $\mathcal{S}$ is continuous, we necessarily deduce that $c_t = -t\tilde{c}$ for all $t > 0$ for some constant $\tilde{c} \in \mathbb{R}$.

The identity $\mathcal{S}(t)\mathbf{u} = \mathbf{u} - t\tilde{c}\mathbb{1}$, for all $t \geqslant 0$, implies that $\mathbf{u}$ is a viscosity solution of (15) with $\tilde{c}$ in place of c, see Proposition 2.6. But then $\tilde{c} = c$ in view of Proposition 2.11 and the statement is proved. □

3. The Aubry set

In this section we start our qualitative analysis on the critical weakly coupled system

$$H_i(x, Du_i) + \big(B(x)\mathbf{u}(x)\big)_i = 0 \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \ldots, m\}. \tag{16}$$

Here and in the remainder of the paper we assume that the critical value c defined via (13) is equal to 0. This renormalization is always possible by replacing each

H_i with $H_i - c$. Solutions, subsolutions and supersolutions of the weakly coupled system (16) will be termed *critical* in the sequel. The family of critical subsolutions, we recall, is denoted by $\mathcal{H}(0)$.

Our qualitative analysis on the critical weakly coupled system is based on the notion of *Mañé matrix*, defined in analogy with that of the Mañé potential.

Definition 3.1. *For all $(x,y,i,j) \in \mathbb{T}^N \times \mathbb{T}^N \times \{1,\dots,m\} \times \{1,\dots,m\}$, we define*

$$\Phi_{i,j}(x,y) = \sup_{\mathbf{v}\in\mathcal{H}(0)} v_i(y) - v_j(x).$$

The following properties hold:

Proposition 3.2. *The Mañé matrix verifies the following properties:*

(i) it is everywhere finite and Lipschtiz continuous;

(ii) $\Phi_{\cdot,j}(y,\cdot) \in \mathcal{H}(0) \quad$ for every $(y,j) \in \mathbb{T}^N \times \{1,\dots,m\}$;

(iii) for every $(y,j) \in \mathbb{T}^N \times \{1,\dots,m\}$ and $\mathbf{v} \in \mathcal{H}(0)$,

$$\mathbf{v} - v_j(y)\mathbb{1} \leqslant \Phi_{\cdot,j}(y,\cdot) \qquad \text{in } \mathbb{T}^N,$$

namely $\Phi_{\cdot,j}(y,\cdot)$ is the maximal critical subsolution whose j–th component vanishes at y;

(iv) the entries of the Mañé matrix are linked by the following triangular inequality:

$$\Phi_{i,k}(x,z) \leqslant \Phi_{j,k}(x,y) + \Phi_{i,j}(y,z)$$

for every $i,j,k \in \{1,\dots,m\}$ and $x,y,z \in \mathbb{T}^N$.

Proof. The fact that the Mañé matrix is well defined directly follows from Proposition 2.2. Lipschitz continuity comes from the equi–Lipschitz character of critical subsolutions.

The second assertion comes from the fact that $\Phi_{\cdot,j}(y,\cdot)$ is, for every fixed (j,y), a supremum of critical subsolutions, hence itself a critical subsolution by Proposition 1.7.

The third point is a direct consequence of the definition.

The last point comes from the fact that $\Phi_{\cdot,j}(y,\cdot)$ is the greatest subsolution whose j–th component vanishes at y. Since $\Phi_{\cdot,k}(x,\cdot) - \Phi_{j,k}(x,y)\mathbb{1}$ is a subsolution whose j–th component vanishes at y we obtain that

$$\Phi_{\cdot,k}(x,\cdot) - \Phi_{j,k}(x,y)\mathbb{1} \leqslant \Phi_{\cdot,j}(y,\cdot),$$

which is the triangular inequality to be proved. □

As in the case of a single critical equation, the Mañé vectors are "almost" critical solutions, in the sense precised below:

Proposition 3.3. *Let $y_0 \in \mathbb{T}^N$ and $i_0 \in \{1,\dots,m\}$. Then the function $\mathbf{u} = \Phi_{\cdot,i_0}(y_0,\cdot)$ satisfies*

$$H_{i_0}(x, Du_{i_0}) + \big(B(x)\mathbf{u}(x)\big)_{i_0} = 0 \quad \text{in } \mathbb{T}^N \setminus \{y_0\}.$$

and

$$H_i(x, Du_i) + \big(B(x)\mathbf{u}(x)\big)_i = 0 \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \neq i_0$$

in the viscosity sense.

Proof. We argue by contradiction, following the classical argument of [18] for the classical Mañé potential.

Let (i,y) be such that either $i \neq i_0$ or $y \neq y_0$. Let us assume that the viscosity supersolution condition is violated at (i,y). This means that there exists a C^1 function ψ such that $\psi(x) \leqslant \Phi_{i,i_0}(y_0,x)$ for all x, with equality if and only if $x = y$, and

$$H_i\big(x, D\psi(y)\big) + \big(B(y)\Phi_{\cdot,i_0}(y_0,y)\big)_i < 0.$$

Since ψ is C^1, and $B(\cdot)$ and $\Phi_{\cdot,i_0}(x_0,\cdot)$ are continuous, it is clear that this strict inequality continues to hold in a neighborhood of y. We infer that it is possible to find $\varepsilon > 0$ small enough such that the function $w_i := \max\{\Phi_{i,i_0}(y_0,\cdot), \psi + \varepsilon\}$ verifies

$$H_i\big(x, Dw_i(x)\big) + \big(B(x)\mathbf{w}(x)\big)_i \leqslant 0 \qquad \text{for a.e. } x \in \mathbb{T}^N,$$

where $\mathbf{w}$ is the vector whose i–th coordinate is w_i and whose other coordinates are those of $\Phi_{\cdot,i_0}(y_0,\cdot)$. In the case when $i = i_0$ and $y \neq y_0$, we choose $\varepsilon > 0$ small enough in such a way that $w_i(y_0) = \Phi_{i,i_0}(y_0,y_0) = 0$. Moreover, for every $j \neq i$,

$$H_j\big(x, Dw_j(x)\big) + \big(B(x)\mathbf{w}(x)\big)_j \leqslant 0 \qquad \text{for a.e. } x \in \mathbb{T}^N,$$

as it is easily seen from the fact that $b_{ji}(\cdot) \leqslant 0$ in $\mathbb{T}^N$ and $w_i \geqslant \Phi_{i,i_0}(y_0,\cdot)$.

We have thus shown that $\mathbf{w}$ is a critical subsolution with $w_{i_0}(y_0) = 0$, $\mathbf{w} \geqslant \Phi_{i,i_0}(y_0,\cdot)$ and $\mathbf{w} \not\equiv \Phi_{i,i_0}(y_0,\cdot)$, thus contradicting the maximality of $\Phi_{i,i_0}(y_0,\cdot)$ amongst subsolutions whose i_0–th coordinate vanishes at y_0. □

Next, we show a strong invariance property enjoyed by the rows of the Mañé matrix.

Proposition 3.4. *Let $i,j \in \{1,\dots,m\}$ and $y \in \mathbb{T}^N$. If $\Phi_{\cdot,i}(y,\cdot)$ is a critical solution on $\mathbb{T}^N$, then $\Phi_{\cdot,j}(y,\cdot)$ is too.*

Proof. Let us set $\mathbf{v} := \Phi_{\cdot,j}(y,\cdot)$ and $\mathbf{u} := \Phi_{\cdot,i}(y,\cdot) + \Phi_{i,j}(y,y)\,\mathbb{1}$. In view of Proposition 3.3, we only need to show that

$$H_j(y,p) + \big(B(y)\mathbf{v}(y)\big)_j \geqslant 0 \quad \text{for every } p \in D^- v_j(y).$$

According to Proposition 3.2, $\mathbf{v} \leqslant \mathbf{u}$ in $\mathbb{T}^N$ and $v_i(y) = u_i(y)$. The functions $\mathbf{v}$ and $\mathbf{u}$ being respectively a critical subsolution and a solution, we can apply Proposition 2.3 to infer that $\mathbf{v}(y) = \mathbf{u}(y)$. This also implies that $D^- v_j(y) \subseteq D^- u_j(y)$. Exploiting again the fact that $\mathbf{u}$ is a critical solution we finally get

$$0 \leqslant H_j(y,p) + \big(B(y)\mathbf{u}(y)\big)_j = H_j(y,p) + \big(B(y)\mathbf{v}(y)\big)_j \quad \text{for every } p \in D^- v_j(y).$$

□

In view of the previous proposition, the following definition is well posed:

Definition 3.5. *The Aubry set $\mathcal{A}$ for the weakly coupled system* (16) *is the set defined as*

$$\mathcal{A} = \big\{ y \in \mathbb{T}^N \,:\, \Phi_{\cdot,i}(y,\cdot) \text{ is a critical solution} \big\},$$

where i is any fixed index in $\{1,\dots,m\}$.

By the continuity of the Mañé matrix and the stability of the notion of viscosity solution, it is easily seen that $\mathcal{A}$ is closed. The analysis we are about to present will show that the Aubry set is nonempty: as in the corresponding critical scalar case, we will see that $\mathcal{A}$ is the set where the obstruction to the existence of globally strict critical subsolutions concentrates.

Definition 3.6. *Let $\mathbf{v} \in \mathcal{H}(0)$. We will say that v_i is* strict at $y \in \mathbb{T}^N$ *if there exist an open neighborhood V of y and $\delta > 0$ such that*

$$H_i\big(x, Dv_i(x)\big) + \big(B(x)\mathbf{v}(x)\big)_i < -\delta \quad \textit{for a.e. } x \in V.$$

We will say that v_i is strict in an open subset U of $\mathbb{T}^N$ if it is strict at y for every $y \in U$.

We start by establishing an auxiliary result that will be needed in the sequel.

Lemma 3.7. *Let $\mathbf{w} \in \mathcal{H}(0)$ such that w_i is strict at $y \in \mathbb{T}^N$. Then there exists $\widetilde{\mathbf{w}} \in \mathcal{H}(0)$ such that $\widetilde{w}_i$ is C^∞ and strict in a neighborhood of y.*

Proof. By hypothesis, there exist $r > 0$ and $\delta > 0$ such that

$$H_i\big(x, Dw_i(x)\big) + \big(B(x)\mathbf{w}(x)\big)_i < -\delta \quad \text{for a.e. } x \in B_{2r}(y).$$

Let $\phi : \mathbb{T}^N \to [0,1]$ be a C^∞–function, compactly supported in $B_r(y)$ and such that $\phi \equiv 1$ in $B_{r/2}(y)$. Let us denote by κ a Lispchitz constant for the critical subsolutions and by ω a continuity modulus of H_i in $\mathbb{T}^N \times B_R$ for some fixed $R > \kappa + \|D\phi\|_\infty$. Let $(\rho_n)_n$ be a sequence of standard mollifiers on $\mathbb{R}^N$ and define

$$\psi_n(x) = (\rho_n * w_i)(x) + \|\rho_n * w_i - w_i\|_\infty, \qquad x \in \mathbb{T}^N.$$

Note that $\psi_n \geqslant w_i$ in $\mathbb{T}^N$ for every $n \in \mathbb{N}$ and

$$d_n := \|\psi_n - w_i\|_\infty \to 0 \qquad \text{as } n \to +\infty.$$

Up to neglecting the first terms, we furthermore assume that all the d_n are less than 1. For every $n \in \mathbb{N}$, we define a function $\widetilde{\mathbf{w}}^n \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$ by setting

$$\widetilde{w}^n_j(x) = w_j(x) \quad \text{if } j \neq i, \qquad \widetilde{w}^n_i(x) = \phi(x)\psi_n(x) + \big(1 - \phi(x)\big)w_i(x)$$

for every $x \in \mathbb{T}^N$. It is apparent by the definition that $\widetilde{w}^n_i$ is of class C^∞ in $B_{r/2}(y)$. Moreover the functions $(\widetilde{w}^n_i)_n$, and hence the $(\widetilde{\mathbf{w}}^n)_n$, are equi–Lipschitz. Indeed, for almost every $x \in \mathbb{T}^N$,

$$D\widetilde{w}^n_i(x) = \phi(x)D\psi_n(x) + \big(1 - \phi(x)\big)Dw_i(x) + \big(\psi_n(x) - w_i(x)\big)D\phi(x) \tag{17}$$

that is $\|D\widetilde{w}^n_i\|_\infty \leqslant \kappa + \|D\phi\|_\infty$. We want to show that n can be chosen sufficiently large in such a way that $\widetilde{\mathbf{w}}^n \in \mathcal{H}(0)$ and

$$H_i\big(x, D\widetilde{w}^n_i(x)\big) + \big(B(x)\widetilde{\mathbf{w}}^n(x)\big)_i < -\frac{2}{3}\,\delta \qquad \text{for a.e. } x \in B_r(y). \tag{18}$$

We first note that, since $\widetilde{w}^n_i \geqslant w_i$ and $b_{ji} \leqslant 0$ in $\mathbb{T}^N$ for every $j \neq i$, we have

$$H_j(x, D\widetilde{w}^n_j(x)) + \big(B(x)\widetilde{\mathbf{w}}^n(x)\big)_j \leqslant 0 \quad \text{in } \mathbb{T}^N \qquad \text{for every } j \neq i. \tag{19}$$

Moreover, since $\widetilde{\mathbf{w}}^n$ agrees with $\mathbf{w}$ outside $B_r(y)$, in order to show that $\widetilde{\mathbf{w}}^n$ satisfies (19) also for $j = i$, it will be enough, by the convexity of H_i, to prove (18).

To this aim, we start by noticing that

$$H_i\big(x, D\widetilde{w}^n_i(x)\big) \leqslant \phi(x)H_i\big(x, D\psi_n(x)\big) + \big(1 - \phi(x)\big)H_i\big(x, Dw_i(x)\big) + \omega\left(d_n \|D\phi\|_\infty\right) \tag{20}$$

for almost every $x \in \mathbb{T}^N$, in view of (17) and of the convexity of H_i. By Jensen's inequality, for every $n > 1/r$ and every $x \in B_r$ we have

$$\begin{aligned} H_i\big(x, D\psi_n(x)\big) &= H_i\left(x, \int_{B_{1/n}} Dw_i(x-y)\rho_n(y)\,\mathrm{d}y\right) \\ &\leqslant \int_{B_{1/n}} H_i\big(x, Dw_i(x-y)\big)\rho_n(y)\,\mathrm{d}y \\ &\leqslant \omega(1/n) + \int_{B_{1/n}} H_i\big(x-y, Dw_i(x-y)\big)\rho_n(y)\,\mathrm{d}y \\ &\leqslant -\int_{B_{1/n}} \big(B(x-y)\mathbf{w}(x-y)\big)_i\,\rho_n(y)\,\mathrm{d}y - \delta + \omega(1/n) \\ &\leqslant -\big(B(x)\widetilde{\mathbf{w}}^n(x)\big)_i - \delta + \omega(1/n) + \varepsilon_n, \end{aligned} \tag{21}$$

where

$$\varepsilon_n := \sup_{|z|\leqslant 1/n} \left\| \big(B(\cdot+z)\mathbf{w}(\cdot+z) - B(\cdot)\widetilde{\mathbf{w}}^n(\cdot)\big)_i \right\|_\infty.$$

Since $\widetilde{\mathbf{w}}^n \rightrightarrows \mathbf{w}$ in $\mathbb{T}^N$ and all these functions are equi–Lipschitz, it is easily seen that $\lim_n \varepsilon_n = 0$. Furthermore

$$H_i\big(x, Dw_i(x)\big) \leqslant -\big(B(x)\widetilde{\mathbf{w}}^n(x)\big)_i - \delta + \varepsilon_n \quad \text{for a.e. } x \in B_r(y). \tag{22}$$

We now choose $n > 1/r$ sufficiently large such that

$$\omega\left(d_n \|D\phi\|_\infty\right) + \omega(1/n) + \varepsilon_n < \frac{\delta}{6}$$

and plug (21) and (22) into (20) to finally get (18). The assertion follows by setting $\widetilde{\mathbf{w}} := \widetilde{\mathbf{w}}^n$ for such an index n. ☐

The next proposition shows that the i–th component of any critical subsolution fulfills the supersolution test on $\mathcal{A}$.

Proposition 3.8. *Let $y \in \mathcal{A}$. Then, for every $i \in \{1, \dots, m\}$ and $\mathbf{w} \in \mathcal{H}(0)$,*

$$H_i(y,p) + \big(B(y)\mathbf{w}(y)\big)_i = 0 \qquad \text{for every } p \in D^- w_i(y). \tag{23}$$

Proof. Pick $\mathbf{w} \in \mathcal{H}(0)$ and set $\mathbf{u} = \Phi_{\cdot,i}(y,\cdot) + w_i(y)\mathbb{1}$. According to Proposition 3.2, $\mathbf{w} \leqslant \mathbf{u}$ and, by definition of $\mathbf{u}$, $w_i(y) = u_i(y)$, in particular $D^- w_i(y) \subseteq D^- u_i(y)$. Now we exploit the fact that $\mathbf{u}$ and $\mathbf{w}$ are a critical solution and subsolution, respectively: from Proposition 2.3 we infer that $\mathbf{w}(y) = \mathbf{u}(y)$, while Proposition 1.6 implies

$$0 \geqslant H_i(y,p) + \big(B(y)\mathbf{w}(y)\big)_i = H_i(y,p) + \big(B(y)\mathbf{u}(y)\big)_i \geqslant 0 \quad \forall p \in D^- w_i(y).$$

Hence all the inequalities must be equalities and the statement follows. ☐

A converse of this result is given by the following

Proposition 3.9. *Let $i \in \{1, \dots, m\}$. The following facts are equivalent:*

(i) $y \notin \mathcal{A}$;

(ii) *there exists $\mathbf{w} \in \mathcal{H}(0)$ such that w_i is strict at y.*

Proof. Let us assume *(i)*. Since $y \notin \mathcal{A}$, the supersolution test for $\Phi_{\cdot,i}(y,\cdot)$ is violated at (i,y). This means that there exists a C^1 function ψ such that $\psi(x) \leqslant \Phi_{i,i}(y,x)$ for all x, with equality if and only if $x = y$, and

$$H_i\big(x, D\psi(y)\big) + \big(B(y)\Phi_{\cdot,i}(y,y)\big)_i < 0.$$

We define a function $\mathbf{w} \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$ by setting

$$w_i(\cdot) = \max\{\Phi_{i,i}(y,\cdot), \psi + \varepsilon\}, \qquad w_j(\cdot) = \Phi_{j,i}(y,\cdot) \quad \text{for } j \neq i.$$

Arguing as in the proof of Proposition 3.3 we see that it is possible to choose $\varepsilon > 0$ in a such a way that $\mathbf{w}$ is a critical subsolution. Moreover, since w_i agrees with $\psi+\varepsilon$ in a neighborhood of y, there exist $\delta > 0$ and an open neighborhood W of y such that w_i is of class C^1 in W and

$$H_i\big(x, Dw_i(x)\big) + \big(B(x)\mathbf{w}(x)\big)_i < -\delta \quad \text{for every } x \in W.$$

Conversely, let assume *(ii)*. According to Lemma 3.7, there exists $\widetilde{\mathbf{w}} \in \mathcal{H}(0)$ such that $\widetilde{w}_i$ is smooth and strict in a neighborhood of y, in particular

$$H_i\big(y, D\widetilde{w}_i(y)\big) + \big(B(y)\widetilde{\mathbf{w}}(y)\big)_i < 0.$$

In view of Proposition 3.8 we conclude that $y \notin \mathcal{A}$. □

Remark 3.10. Proposition 3.8 expresses the fact, roughly speaking, that the i–th component of a critical subsolution cannot be strict at y. However, since the supersolution test (23) is void when $D^-u_i(y)$ is empty, this fact cannot be directly used to prove the equivalence stated in Proposition 3.9. This is precisely the reason why we needed the regularization Lemma 3.7.

We proceed by proving a global version of the previous proposition. We give a definition first.

Definition 3.11. *Let $\mathbf{v} \in \mathcal{H}(0)$. We will say that $\mathbf{v}$ is* strict at y *if v_i is strict at y for every $i \in \{1, \dots, m\}$. We will say that $\mathbf{v}$ is strict in an open subset U of $\mathbb{T}^N$ if it is strict at y for every $y \in U$.*

Theorem 3.12. *There exists $\mathbf{v} \in \mathcal{H}(0)$ which is strict in $\mathbb{T}^N \setminus \mathcal{A}$. In particular, the Aubry set $\mathcal{A}$ is closed and nonempty.*

Proof. Fix $i \in \{1, \dots, m\}$. We first construct a critical subsolution $\mathbf{v}^i$ whose i–th component is strict in $\mathbb{T}^N \setminus \mathcal{A}$. According to Proposition 3.9, for every $y \in \mathbb{T}^N \setminus \mathcal{A}$ there exist an open neighborhood W_y of y, a critical subsolution $\mathbf{w}^y$ and $\delta_y > 0$ such that

$$H_i\big(x, Dw_i^y(x)\big) + \big(B(x)\mathbf{w}^y(x)\big)_i < -\delta_y \quad \text{for a.e. } x \in W_y \tag{24}$$

The family $\{W_y : y \in \mathbb{T}^N \setminus \mathcal{A}\}$ is an open covering of $\mathbb{T}^N \setminus \mathcal{A}$, from which we can extract a countable covering $(W_n)_n$ of $\mathbb{T}^N \setminus \mathcal{A}$. For each $n \in \mathbb{N}$, let us denote by $(\mathbf{w}^n, \delta_n)$ the corresponding pair in $\mathcal{H}(0) \times (0, +\infty)$ that satisfies (24) in W_n. Up to subtracting to each critical subsolution $\mathbf{w}^n$ a vector of the form $k_n\mathbb{1}$, we can moreover assume that $w_1^n(0) = 0$. Hence the functions $\mathbf{w}^n$ are componentwise equi–Lipschitz and equi–bounded in view of Proposition 2.2, in particular the function

$$\mathbf{v}^i(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}\, \mathbf{w}^n(x), \qquad x \in \mathbb{T}^N$$

is well defined and belongs to $\big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$. By convexity of the Hamiltonians, for almost every $x \in \mathbb{T}^N$ we get

$$H_i\big(x, Dv_i^i(x)\big) + \big(B(x)\mathbf{v}^i(x)\big)_i \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n}\Big(H_i\big(x, Dw_i^n(x)\big) + \big(B(x)\mathbf{w}^n(x)\big)_i\Big) \leqslant 0.$$

Moreover, the above inequalities hold with $-\delta_k/2^k$ in place of 0 almost everywhere in W_k, for every $k \in \mathbb{N}$. This shows that $\mathbf{v}^i$ is a critical subsolution, strict in $\mathbb{T}^N \setminus \mathcal{A}$. Now set

$$\mathbf{v}(x) = \sum_{i=1}^{m} \frac{1}{m}\mathbf{v}^i(x), \qquad x \in \mathbb{T}^N.$$

A similar argument shows that $\mathbf{v}$ is a critical subsolution that satisfies the assertion.

If $\mathcal{A} = \varnothing$, by compactness we would have $\mathcal{H}(-\delta) \neq \varnothing$ for some $\delta > 0$, contradicting the definition of the critical value $c = 0$. $\square$

In view of Proposition 3.9, we have the following characterization:

Theorem 3.13. *Let $y \in \mathbb{T}^N$. The following are equivalent facts:*

(i) $y \notin \mathcal{A}$;

(ii) there exists $\mathbf{w} \in \mathcal{H}(0)$ *which is strict at* y;

(iii) there exists $\mathbf{w} \in \mathcal{H}(0)$ *and* $i \in \{1, \dots, m\}$ *such that* w_i *is strict at* y.

We end this section by extending to weakly coupled systems a result which is well known in the case of a single critical equation.

Proposition 3.14. $$\mathcal{A} = \bigcap_{\mathbf{w} \in \mathcal{H}(0)} \big\{ y \in \mathbb{T}^N \,:\, \big(\mathcal{S}(t)\mathbf{w}\big)(y) = \mathbf{w}(y) \quad \textit{for every } t > 0 \big\}.$$

Proof. Let us denote by $\mathcal{A}'$ the set appearing at the right–hand side of the above equality. Fix a point $y \in \mathcal{A}$ and let $\mathbf{w}$ be any critical subsolution. For every fixed index $i \in \{1, \dots, m\}$, the function $\mathbf{u}^i = \Phi_{\cdot,i}(y, \cdot) + w_i(y)\mathbb{1}$ satisfies $\mathbf{w} \leqslant \mathbf{u}^i$ in $\mathbb{T}^N$ and $w_i(y) = u_i(y)$. Moreover, $\mathbf{u}^i$ is a critical solution, hence it is a fixed point for the semigroup $\mathcal{S}(t)$ by Proposition 2.6. By monotonicity of the semigroup, we have

$$w_i(y) \leqslant \big(\mathcal{S}(t)\mathbf{w}\big)_i(y) \leqslant \big(\mathcal{S}(t)\mathbf{u}^i\big)_i(y) = u_i^i(y) \quad \text{for every } t > 0,$$

hence all the inequalities must be equalities, in particular $\big(\mathcal{S}(t)\mathbf{w}\big)_i(y) = w_i(y)$ for every $t > 0$. This being true for every $i \in \{1, \dots, m\}$ and $\mathbf{w} \in \mathcal{H}(0)$, we conclude that $y \in \mathcal{A}'$.

To prove the converse inclusion, we take a critical subsolution $\mathbf{v}$ which is strict outside $\mathcal{A}$. According to Proposition A.3, for every $y \notin \mathcal{A}$ there exists $t_y > 0$ such that $\big(\mathcal{S}(t_y)\mathbf{v}\big)(y) > \mathbf{v}(y)$, that is $y \notin \mathcal{A}'$. $\square$

4. Regularization

The aim of this section is to show how a strict critical subsolution can be regularized outside the Aubry set. In the case of a single critical equation, it is known that such procedure can be performed in such a way that the output is a strict critical subsolution which is, in addition, of class C^1 on the whole torus, see [3, 19, 20]. This result holds for Hamiltonians that are locally Lipschitz in (x, p) and strictly convex in p and the proof relies on the following two facts: first, any critical subsolution

is differentiable on the Aubry set and, second, its gradient is independent of the specific subsolution chosen. This latter rigidity property holds for weakly couples systems too, as we will show at the end of the current section. What prevents us to extend to systems the existence of C^1 strict critical subsolutions is the lack of information on differentiability properties of critical subsolutions on the Aubry set.

We first deal with the regularization issue. The tools are not new and are mainly borrowed from [19, 20]. However, we provide a proof for the reader's convenience.

We start with a local regularization argument.

Lemma 4.1. *Let $\mathbf{u} \in \mathcal{H}(0)$ and assume that, for some $r > 0$, $\delta > 0$ and $y \in \mathbb{T}^N \setminus \mathcal{A}$ and for every $i \in \{1, \dots, m\}$,*

$$H_i\big(x, Du_i(x)\big) + \big(B(x)\mathbf{u}(x)\big)_i < -\delta \qquad \textit{for a.e. } x \in B_{2r}(y).$$

Then, for every $\varepsilon > 0$, there exists $\mathbf{u}^\varepsilon \in \mathcal{H}(0)$ such that

(i) $\|\mathbf{u}^\varepsilon - \mathbf{u}\|_\infty < \varepsilon$;

(ii) $\mathbf{u}^\varepsilon = \mathbf{u}$ *in* $\mathbb{T}^N \setminus B_r(y)$;

(iii) $\mathbf{u}^\varepsilon$ *is of class C^∞ in $B_{r/2}(y)$ and satisfies*

$$H_i\big(x, Du_i^\varepsilon(x)\big) + \big(B(x)\mathbf{u}^\varepsilon(x)\big)_i < -\frac{2}{3}\,\delta \qquad \textit{for every } x \in B_{r/2}(y). \tag{25}$$

Proof. Let $\phi : \mathbb{T}^N \to [0, 1]$ be C^∞ function, compactly supported in $B_r(y)$ and such that $\phi \equiv 1$ in $B_{r/2}(y)$. Let $(\rho_n)_n$ be a sequence of standard mollifiers on $\mathbb{R}^N$. For every $n \in \mathbb{N}$, we define a function $\mathbf{w}^n \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$ by setting

$$w_i^n(x) = \phi(x)(\rho_n * u_i)(x) + \big(1 - \phi(x)\big)u_i(x) \qquad \text{for every } x \in \mathbb{T}^N \text{ and } i \in \{1, \dots, m\}.$$

It is apparent by the definition that $\mathbf{w}^n$ is of class C^∞ in $B_{r/2}(y)$ and agrees with $\mathbf{u}$ outside $B_r(y)$. Arguing as in the proof of Lemma 3.7, we see that it is possible to choose n large enough in such a way that $\mathbf{w}^n$ is a critical subsolution and satisfies (25). Since $\mathbf{w}^n \rightrightarrows \mathbf{u}$ in $\mathbb{T}^N$, the assertion follows by seting $\mathbf{u}^\varepsilon := \mathbf{w}^n$ for a sufficiently large n. □

We now prove the announced regularization result.

Theorem 4.2. *There exists a critical subsolution which is strict and C^∞ in $\mathbb{T}^n \setminus \mathcal{A}$. More precisely, for every critical subsolution $\mathbf{v}$ which is strict in $\mathbb{T}^N \setminus \mathcal{A}$ and for every $\varepsilon > 0$, there exists $\mathbf{v}^\varepsilon \in \mathcal{H}(0)$ such that*

(i) $\|\mathbf{v}^\varepsilon - \mathbf{v}\|_\infty < \varepsilon$;

(ii) $\mathbf{v}^\varepsilon = \mathbf{v}$ *on* $\mathcal{A}$;

(iii) $\mathbf{v}^\varepsilon$ *is C^∞ and strict in $\mathbb{T}^n \setminus \mathcal{A}$.*

Moreover, the set of such smooth and strict subsolutions is dense in $\mathcal{H}(0)$.

Proof. We first show how to regularize a subsolution which is strict outside the Aubry set. Let $\mathbf{v}$ be such a subsolution (given by Theorem 3.12) and fix $\varepsilon > 0$. Since $\mathbf{v}$ is strict in $\mathbb{T}^N \setminus \mathcal{A}$, there exists a continuous and non–negative function $\delta : \mathbb{T}^N \to \mathbb{R}$ with $\delta^{-1}(\{0\}) = \mathcal{A}$ such that

$$H_i(x, Dv_i) + \big(B(x)\mathbf{v}(x)\big)_i \leqslant -\delta(x) \quad \text{in } \mathbb{T}^N$$

for every $i \in \{1, \dots, m\}$. Clearly, it is not restrictive to assume that the inequality $\delta(x) < \varepsilon$ holds for every $x \in \mathbb{T}^N$. In view of Lemma 4.1, we can find a locally finite

covering $(U_n)_n$ of $\mathbb{T}^N \setminus \mathcal{A}$ by open sets compactly contained in $\mathbb{T}^N \setminus \mathcal{A}$ and a sequence $(\mathbf{u}^n)_n$ of critical subsolutions such that each $\mathbf{u}^n$ is C^∞ in U_n and satisfies

$$\begin{aligned} H_i(x, Du_i^n) + \big(B(x)\mathbf{u}^n(x)\big)_i \leqslant -\frac{2}{3}\,\delta(x) \qquad &\text{for every } x \in U_n, \\ |\mathbf{u}^n(x) - \mathbf{v}(x)| \leqslant \delta(x) \qquad &\text{for every } x \in \mathbb{T}^N. \end{aligned} \tag{26}$$

Set

$$\delta_n := \inf_{x \in U_n} \delta(x) \qquad \text{for every } n \in \mathbb{N}$$

and choose a sequence $(\eta_n)_n$ in $(0,1)$ such that, for every $x \in \mathbb{T}^N$ and $n \in \mathbb{N}$, the following holds:

$$|H(x,p) - H(x,p')| < \frac{\delta_n}{6} \qquad \text{for all } p, p' \in B_{\kappa+1} \text{ with } |p - p'| < \eta_n, \tag{27}$$

where κ denotes a common Lipschitz constant for the critical subsolutions, in particular for all the $\mathbf{u}^n$. Last, take a smooth partition of unity $(\varphi_n)_n$ subordinate to $(U_n)_n$ and choose the functions $\mathbf{u}^n$ in such a way that the quantities $\|\mathbf{u}^n - \mathbf{v}\|_\infty$, which can be be made as small as desired, satisfy

$$\sum_{\substack{k \in \mathbb{N} \\ U_k \cap U_n \neq \varnothing}} \|\mathbf{u}^k - \mathbf{v}\|_\infty \, \|D\varphi_k\|_\infty < \eta_n \qquad \text{for every } n \in \mathbb{N}. \tag{28}$$

That is always possible since the covering $(U_n)_n$ is locally finite.

We now define $\mathbf{v}^\varepsilon : \mathbb{T}^N \to \mathbb{R}^m$ by setting

$$\mathbf{v}^\varepsilon(x) = \sum_{n=1}^{\infty} \varphi_n(x)\mathbf{u}^n(x) \quad \text{in } \mathbb{T}^N \setminus \mathcal{A} \qquad \text{and} \qquad \mathbf{v}^\varepsilon(x) = \mathbf{v}(x) \quad \text{on } \mathcal{A}.$$

By definition, $\mathbf{v}^\varepsilon$ satisfies assertion *(ii)* and is C^∞ in $\mathbb{T}^N \setminus \mathcal{A}$. From (26) we infer that $|\mathbf{v}^\varepsilon(x) - \mathbf{v}(x)| \leqslant \delta(x)$ in $\mathbb{T}^N \setminus \mathcal{A}$, which shows at once that $\mathbf{v}^\varepsilon$ is continuous in $\mathbb{T}^N$ and that it satisfies assertion *(i)*. Moreover, by taking into account (28) and the fact that $\sum D\varphi_k \equiv 0$, one obtains, for every $x \in U_n$ and $i \in \{1, \ldots, m\}$, that

$$\Big| Dv_i^\varepsilon(x) - \sum_{\substack{k \in \mathbb{N} \\ U_k \cap U_n \neq \varnothing}} \varphi_k(x) Du_i^k(x) \Big| = \Big| \sum_{\substack{k \in \mathbb{N} \\ U_k \cap U_n \neq \varnothing}} \big(u_i^k(x) - v(x)\big) D\varphi_k(x) \Big| < \eta_n, \tag{29}$$

in particular

$$|Dv_i^\varepsilon(x)| \leqslant \eta_n + \sum_{\substack{k \in \mathbb{N} \\ U_k \cap U_n \neq \varnothing}} \varphi_k(x) |Du_i^k(x)| \leqslant 1 + \kappa.$$

We infer that $\mathbf{v}^\varepsilon$ is Lipschitz–continuous in $\mathbb{T}^N$. In order to prove that $\mathbf{v}^\varepsilon$ is a critical subsolution and is strict in $\mathbb{T}^N \setminus \mathcal{A}$, it will be enough to show that

$$H_i(x, Dv_i^\varepsilon(x)) + \big(B(x)\mathbf{v}^\varepsilon(x)\big)_i \leqslant -\frac{\delta(x)}{2} \qquad \text{for a.e. } x \in \mathbb{T}^N,$$

for all $i \in \{1, \ldots, m\}$.

Since $D\mathbf{v}^\varepsilon(x) = D\mathbf{v}(x)$ for almost every $x \in \mathcal{A}$, it suffices to establish the claim in the complementary of $\mathcal{A}$. To this aim, by recalling the definition of η_n and by

making use of (29) and of Jensen inequality, we get that, for every $x \in U_n$ and $i \in \{1, \dots, m\}$,

$$\begin{aligned}
H_i\big(x, Dv_i^\varepsilon(x)\big) + \big(B(x)\mathbf{v}^\varepsilon(x)\big)_i &\leqslant H_i\Big(x, \sum_{\substack{k\in\mathbb{N}\\ U_k\cap U_n\neq\varnothing}} \varphi_k(x) Du_i^k(x)\Big) + \frac{\delta_n}{6} \\
&\quad + \sum_{\substack{k\in\mathbb{N}\\ U_k\cap U_n\neq\varnothing}} \varphi_k(x)\big(B(x)\mathbf{u}^k(x)\big)_i \\
&\leqslant \sum_{\substack{k\in\mathbb{N}\\ U_k\cap U_n\neq\varnothing}} \varphi_k(x)\Big(H_i\big(x, Du_i^k(x)\big) + \big(B(x)\mathbf{u}^k(x)\big)_i\Big) + \frac{\delta_n}{6} \\
&< -\frac{2}{3}\,\delta(x) + \frac{\delta_n}{6} \leqslant -\frac{\delta(x)}{2}.
\end{aligned}$$

This concludes the proof of the first part of the statement.

For the density, let $\mathbf{u}$ be any critical subsolution. Let $\mathbf{v}$ be a critical subsolution which is strict outside the Aubry set (whose existence is assured by Theorem 3.12). Then, for any $\lambda \in (0, 1)$, the function $(1-\lambda)\mathbf{u} + \lambda\mathbf{v}$ is a subsolution which is strict outside the Aubry set. This subsolution can therefore be regularized using the above procedure, giving a subsolution $\mathbf{w}$ which is strict and smooth outside the Aubry set. Moreover, both these steps can be done in such a way that $\|\mathbf{u} - \mathbf{w}\|_\infty$ is as small as wanted. This establishes the density. $\square$

We now additionally assume the Hamiltonians H_i to be strictly convex in p and derive some further information on the behavior of Clarke's generalized gradients of the critical subsolutions on the Aubry set.

We start with a preliminary lemma.

Lemma 4.3. *Let $y \in \mathcal{A}$ and let $\mathbf{u}^1, \cdots, \mathbf{u}^\ell$ be critical subsolutions. Then, for all $i \in \{1, \dots, m\}$,*

$$\bigcap_{k=1}^{\ell} \partial^c u_i^k(x) \neq \varnothing.$$

Moreover, it contains a vector p_i which is extremal for all the sets $\partial^c u_i^k(x)$ and which satisfies

$$H_i(y, p_i) + \big(B(y)\mathbf{u}^k(y)\big)_i = 0 \qquad \textit{for every } k \in \{1, \dots, \ell\}.$$

Proof. Let $\mathbf{w} = \dfrac{1}{\ell}\sum\limits_{k=1}^{\ell} \mathbf{u}^k \in \mathcal{H}(0)$ and let $p_i \in \partial^c w_i(y)$ be such that

$$H_i(y, p) + \big(B(y)\mathbf{w}(y)\big)_i = 0.$$

Such a p_i must exist because otherwise w_i would be strict at y. Note that, by strict convexity of H_i, the vector p_i must be an extremal point of $\partial^c w_i(x)$, hence it is a reachable gradient of w_i. Let $y_n \to y$ be such that u_i^k is differentiable at y_n for every $k \in \{1, \dots, \ell\}$ and $n \in \mathbb{N}$, and

$$Dw_i(y_n) = \frac{1}{\ell}\sum_{k=1}^{\ell} Du_i^k(y_n) \to p_i.$$

Up to extraction of a subsequence, we can assume that $Du_i^k(y_n) \to q_k$ for all $k \in \{1, \dots, \ell\}$. Then one readily obtains, by Jensen's inequality, that

$$0 = H_i(y, p_i) + \big(B(y)\mathbf{w}(y)\big)_i \leqslant \frac{1}{\ell} \sum_{k=1}^{\ell} \Big(H_i(y, q_k) + \big(B(y)\mathbf{u}^k(y)\big)_i \Big) \leqslant 0.$$

Therefore, all the inequalities $\frac{1}{\ell}\big(H_i(y, q_k) + (B(y)\mathbf{u}^k(y))_i\big) \leqslant 0$ summing to an equality, we deduce, by strict convexity of H_i, that $q_1 = \dots = q_l = p$. Moreover, since

$$H_i(y, q_k) + \big(B(y)\mathbf{u}^k(y)\big)_i = 0 \qquad \text{for every } k \in \{1, \dots, \ell\},$$

and because of the strict convexity of H_i, one sees that p_i is extremal, and thus reachable, for all the u_i^k. □

We now extend the previous result as follows:

Proposition 4.4. *Let $y \in \mathcal{A}$. Then, for each $i \in \{1, \dots, m\}$, there exists a vector $p_i \in \mathbb{R}^N$ which is a reachable gradient of u_i at y for every $\mathbf{u} \in \mathcal{H}(0)$ and which satisfies*

$$H_i(y, p_i) + \big(B(y)\mathbf{u}(y)\big)_i = 0.$$

Proof. For each critical subsolution $\mathbf{u}$, let us denote by $P_i^{\mathbf{u}}$ the set of reachable gradients p of u_i at y that satisfy $H_i(y, p) + \big(B(y)\mathbf{u}(y)\big)_i = 0$. This set is not empty and compact. The proposition amounts to proving that

$$\bigcap_{\mathbf{u} \in \mathcal{H}(0)} P_i^{\mathbf{u}} \neq \varnothing.$$

If this were the case, by compactness we could extract a finite empty intersection. But this would violate the previous lemma. □

5. Rigidity of the Aubry set and Comparison Principle

In this section we establish some interesting properties of the Aubry set and we provide uniqueness results for critical solutions. As we will see, such results will follow rather easily thanks to the information gathered so far.

We start with a remarkable rigidity phenomenon that takes place on the Aubry set.

Theorem 5.1. *Let $y \in \mathcal{A}$ and $i \in \{1, \dots, m\}$. Then*

$$\mathbf{v}(y) = \Phi_{\cdot, i}(y, y) + v_i(y)\mathbb{1} \qquad \textit{for every } \mathbf{v} \in \mathcal{H}(0).$$

In particular, $\quad \mathbf{v}(y) - \mathbf{w}(y) \in \mathbb{R}\mathbb{1}$ *for any* $\mathbf{v}$, $\mathbf{w} \in \mathcal{H}(0)$.

Proof. Take $\mathbf{v} \in \mathcal{H}(0)$ and set $\mathbf{u} := \Phi_{\cdot, i}(y, \cdot) + v_i(y)\mathbb{1}$. According to Proposition 3.2, $\mathbf{u}$ is a critical solution satisfying $\mathbf{v} \leqslant \mathbf{u}$ in $\mathbb{T}^N$ and $v_i(y) = u_i(y)$. By applying Proposition 2.3 with $x_0 := y$ we get the assertion. □

Remark 5.2. On the other hand, the above property does not hold when $y \notin \mathcal{A}$. Indeed, the proof of Lemma 3.7 shows that any critical subsolution $\mathbf{v}$ which is strict at y can be modified in such a way that the output is a critical subsolution all of whose components except one coincide at y with those of $\mathbf{v}$.

Next, we derive a comparison principle for sub and supersolutions of the critical weakly coupled system (16) which generalizes to our setting an analogous result established in [5] for Hamiltonians of a special Eikonal form, see Subsection 6.1 for more details. In particular, we obtain that $\mathcal{A}$ is a uniqueness set for the critical system.

Theorem 5.3. *Let* $\mathbf{v}$, $\mathbf{u} \in \big(\mathrm{C}(\mathbb{T}^N)\big)^m$ *be a sub and a supersolution of the critical weakly coupled system* (16)*, respectively. Assume that*

$$\sum_{i=1}^m \Lambda_i(x) v_i(x) \leqslant \sum_{i=1}^m \Lambda_i(x) u_i(x) \qquad \text{for every } x \in \mathcal{A}, \tag{30}$$

where $\Lambda : \mathcal{A} \to (\mathbb{R}_+)^m$ *is a function satisfying*

$$\sum_{i=1}^m \Lambda_i(x) > 0 \qquad \text{for every } x \in \mathcal{A}.$$

Then

$$\mathbf{v}(x) \leqslant \mathbf{u}(x) \qquad \text{for every } x \in \mathbb{T}^N.$$

In particular, two critical solutions that coincide on $\mathcal{A}$ *coincide on the whole* $\mathbb{T}^N$*.*

Remark 5.4. The above theorem also implies that two critical solutions $\mathbf{u}$ and $\mathbf{v}$ are actually the same if their i–th components coincide on $\mathcal{A}$, for some fixed index $i \in \{1, \dots, m\}$. This is consistent with Theorem 5.1, which assures that this "boundary" condition amounts to requiring that $\mathbf{u} = \mathbf{v}$ on $\mathcal{A}$.

Proof. In view of the density result stated in Theorem 4.2, the critical subsolution $\mathbf{v}$ can be approximated from below by a sequence of critical subsolutions that are, in addition, smooth and strict outside $\mathcal{A}$. Indeed, just pick a sequence $(\mathbf{w}^n)_{n\in\mathbb{N}}$ such that $\|\mathbf{w}^n - \mathbf{v}\|_\infty < n^{-1}$ and then define $\mathbf{v}^n = \mathbf{w}_n - n^{-1}\mathbb{1}$ which then verifies $\mathbf{v}^n \leqslant \mathbf{v}$ and $\|\mathbf{v}^n - \mathbf{v}\|_\infty < 2n^{-1}$. Clearly, each element of the sequence still satisfies the boundary condition (30), hence it is enough to prove the statement by additionally assuming $\mathbf{v}$ smooth and strict in $\mathbb{T}^N \setminus \mathcal{A}$.

Let us set

$$M := \max_{1\leqslant i\leqslant m} \max_{\mathbb{T}^N} (v_i - u_i)$$

and pick a point $x_0 \in \mathbb{T}^N$ where such a maximum is attained. By Proposition 2.3 we know that $\mathbf{v}(x_0) = \mathbf{u}(x_0) + M\mathbb{1}$. If $x_0 \notin \mathcal{A}$, then v_1 would be a smooth subtangent to u_1 at x_0. The function $\mathbf{u}$ being a supersolution, we would have

$$0 \leqslant H_1\big(x_0, Dv_1(x_0)\big) + \big(B(x_0)\mathbf{u}(x)\big)_1 = H_1\big(x_0, Dv_1(x_0)\big) + \big(B(x_0)\mathbf{v}(x)\big)_1,$$

in contrast with the fact that $\mathbf{v}$ is strict in $\mathbb{T}^N \setminus \mathcal{A}$. Hence $x_0 \in \mathcal{A}$ and by (30) we get

$$M \sum_{i=1}^m \Lambda_i(x_0) = \sum_{i=1}^m \Lambda_i(x_0)\left(v_i(x_0) - u_i(x_0)\right) \leqslant 0,$$

i.e. $M \leqslant 0$ as it was to be proved. □

Last, we show that the trace of any critical subsolution on the Aubry set can be extended to the whole torus in such a way that the output is a critical solution.

Theorem 5.5. *For any* $\mathbf{v} \in \mathcal{H}(0)$*, there exists a unique critical solution* $\mathbf{u}$ *such that* $\mathbf{u} = \mathbf{v}$ *on* $\mathcal{A}$*.*

Proof. This assertion is derived as a consequence of Proposition 3.14 by setting

$$\mathbf{u}(x) = \lim_{t\to+\infty} \mathcal{S}(t)v(x) \qquad \text{for every } x \in \mathbb{T}^N.$$

Indeed, $\mathcal{S}(t)\mathbf{v} = \mathbf{v}$ on $\mathcal{A}$ for every $t > 0$. Since the functions $\{\mathcal{S}(t)\mathbf{v} : t > 0\}$ are equi–Lipschitz and non–decreasing with respect to t, we infer that $\mathcal{S}(t)\mathbf{v} \rightrightarrows \mathbf{u}$ in $\mathbb{T}^N$. Last, $\mathbf{u}$ being a fixed point of the semigroup $\mathcal{S}(t)$, we get that $\mathbf{u}$ is a critical solution. □

6. Examples

The critical value and the Aubry set for a weakly coupled system of the kind studied in this paper have, in general, no connections with those of each Hamiltonian, considered individually. This happens also in simple situations, see for instance Example 1.2 in [28]. In this section, we present a couple of examples where more explicit results may be obtained for the critical value and for the Aubry set.

6.1. **The setting of** [5]**.** The first example we propose corresponds to the setting considered in [5]. Assume that all the Hamiltonians are of the form $H_i(x,p) = F_i(x,p) - V_i(x)$, where:

(a) F_i and V_i take non–negative values;
(b) F_i is convex and coercive in p;
(c) $F_i(x,0) = 0$ for all $x \in \mathbb{T}^N$ and $i \in \{1,\ldots,m\}$.

Furthermore, assume that

$$\bigcap_{i=1}^m V_i^{-1}(\{0\}) \neq \varnothing.$$

Under these hypotheses, we claim that the critical value is 0 (whatever the coupling matrix is) and that the Aubry set is nothing but

$$\mathcal{A} = \bigcap_{i=1}^m V_i^{-1}(\{0\}).$$

Indeed, it is easily seen that the null function $\mathbf{u}^0$ always belongs to $\mathcal{H}(0)$ under the first set of hypotheses. Therefore, $\mathcal{H}(0) \neq \varnothing$ and the critical value verifies $c \leqslant 0$. To see that there is actually equality, consider a point $x_0 \in \cap V_i^{-1}(\{0\})$ and any (C^1) function $\mathbf{u}$. An easy consequence of Proposition 1.4 yields that $B(x_0)\mathbf{u}(x_0)$ must have a non–negative entry, say i, hence

$$H_i\big(x_0, Du_i(x_0)\big) + \big(B(x_0)\mathbf{u}(x_0)\big)_i = F_i\big(x_0, Du_i(x_0)\big) + \big(B(x_0)\mathbf{u}(x_0)\big)_i \geqslant 0.$$

Therefore, $\mathbf{u}$ cannot belong to a $\mathcal{H}(-\varepsilon)$ for a positive ε. The same argument can be adapted in the viscosity sense for any (non necessarily C^1) function. Therefore 0 is the critical value.

To prove that $\cap V_i^{-1}(\{0\})$ is the Aubry set, first notice that, for every $y \notin \cap V_i^{-1}(\{0\})$, there exists an index j such that $V_j(y) > 0$. Then the j–th component of the null function $\mathbf{u}^0$ is strict at y. In view of Theorem 3.13 we get the inclusion

$$\mathcal{A} \subseteq \bigcap_{i=1}^m V_i^{-1}(\{0\}).$$

The opposite inclusion is obtained as previously. Take any $\mathbf{u} \in \mathcal{H}(0)$ and $x_0 \in \cap V_i^{-1}(\{0\})$. We will do as if $\mathbf{u}$ is differentiable at x_0, but the argument carries on

in the general case using test functions and the viscosity subsolution property. At x_0 we must have

$$\Big(F_i\big(x_0, Du_i(x_0)\big)\Big)_{i\in\{1,\ldots,m\}} + B(x_0)\mathbf{u}(x_0) \leqslant 0\mathbb{1}.$$

But this is only possible if $Du_i(x_0) = 0$ for all $i \in \{1, \ldots, m\}$ and if $B(x_0)\mathbf{u}(x_0) = 0$. Indeed, otherwise, $B(x_0)\mathbf{u}(x_0)$ will have a positive entry, which is impossible. In particular, the above inequality holds with an equality. Since this happens for any critical subsolution $\mathbf{u}$, we get $x_0 \in \mathcal{A}$ in view of Theorem 3.13. As a byproduct, this also establishes that at any point of $\mathcal{A}$, any critical subsolution must take as value a vector belonging to $\mathbb{R}\,\mathbb{1}$. This is a particular case of Theorem 5.1 (and in this particular form, it also appears in a weaker form in [5]).

6.2. **Commuting Hamiltonians.** In this second example we consider the case when the Hamiltonians are strictly convex and pairwise commute. If the Hamiltonians are of class C^1, that means

$$\{H_i, H_j\}(x, p) := \Big(\frac{\partial H_i}{\partial p}\frac{\partial H_j}{\partial x} - \frac{\partial H_j}{\partial p}\frac{\partial H_i}{\partial x}\Big)(x, p) = 0 \quad \text{in } \mathbb{T}^N \times \mathbb{R}^N$$

for every $i, j \in \{1, \ldots, m\}$. If the Hamiltonians are only continuous, the commutation hypothesis must be expressed in terms of commutation of their Lax–Oleinik semigroup, see [11] for more details. We also make the additional assumption that, individually, all the Hamiltonians have 0 as critical value. Then, we claim that 0 is the critical value of the system as well (whatever the coupling is).

Indeed, it is proved in [11, 30] that the Hamiltonians have the same critical solutions. In particular, there exists a function $u \in \mathrm{Lip}(\mathbb{T}^N)$ satisfying

$$H_i(x, Du) = 0 \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \ldots, m\}$$

in the viscosity sense. Since the coupling is degenerate, we infer that the function $\mathbf{u}^0 = u\mathbb{1}$ is a solution of

$$H_i(x, Du_i^0) + \big(B(x)\mathbf{u}^0(x)\big)_i = 0 \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \ldots, m\}.$$

Therefore, the claim is a direct consequence of Proposition 2.11. Moreover, in this setting, we may localize the Aubry set of the system using those of the individual Hamiltonians. In order to do so, let us recall another result from [11].

Theorem 6.1. *Let $H_1, \cdots, H_m$ be pairwise commuting and strictly convex Hamiltonians, with common critical value equal to 0. Then they have the same Aubry set $\mathcal{A}^*$. Moreover, there exists a common critical subsolution v which is smooth outside $\mathcal{A}^*$ and strict for each Hamiltonian, i.e.*

$$H_i\big(x, Dv(x)\big) < 0 \quad \textit{for every } x \in \mathbb{T}^N \setminus \mathcal{A}^* \textit{ and } i \in \{1, \ldots, m\}.$$

Using this theorem, we easily see that the inclusion $\mathcal{A} \subseteq \mathcal{A}^*$ holds. Indeed, the function $\mathbf{v}(x) := v(x)\mathbb{1}$ is a critical subsolution for the system which is strict outside $\mathcal{A}^*$.

We also note that, as in the previous example, $\mathbf{u}(y) \in \mathbb{R}\,\mathbb{1}$ for every $y \in \mathcal{A}$ and every $\mathbf{u} \in \mathcal{H}(0)$ in view of Theorem 5.1.

A particular case of this example is when all the H_i are equal. In this case we get the more precise statement:

Proposition 6.2. *Let H be a convex Hamiltonian and assume $H_1 = \cdots = H_m = H$. Then $\mathcal{A} = \mathcal{A}^*$. Moreover, all critical solutions of the system are of the form $\mathbf{u} = u\mathbb{1}$ where u is a critical solution of H.*

Proof. The inclusion $\mathcal{A} \subseteq \mathcal{A}^*$ can be proved arguing as above (note that we do not need the strict convexity assumption here). Let us prove the converse statement. Pick $\mathbf{v} \in \mathcal{H}(0)$ and set $v := \max(v_i, \ \ i \in \{1, \ldots, m\})$. We claim that v is a critical subsolution for H. Indeed, let $x \in \mathbb{T}^N$ and $p \in D^+v(x)$. Then $v(x) = v_i(x)$ for some $i \in \{1, \ldots, m\}$. Since $v \geqslant v_i$ with equality at x, we get $p \in D^+v_i(x)$. We now use the fact that $\mathbf{v}$ is a subsolution of the system to get

$$H(x, p) \leqslant H_i(x, p) + \big(B(x)\mathbf{v}(x)\big)_i \leqslant 0, \tag{31}$$

where the first inequality comes from the fact that

$$\big(B(x)\mathbf{v}(x)\big)_i = \sum_{j=1}^{m} b_{ij}(x)v_j(x) \geqslant \sum_{j=1}^{m} b_{ij}(x)v_i(x) = 0,$$

which holds true since $b_{ij}(x) \leqslant 0$ and $v_j(x) \leqslant v_i(x)$ for every $j \neq i$. Let us now assume that $\mathbf{v}$ is strict outside $\mathcal{A}$. Then the right inequality in (31) is strict as soon as $x \notin \mathcal{A}$, yielding that v is a subsolution for H which strict on the complementary of $\mathcal{A}$. This proves that $\mathcal{A}^* \subseteq \mathcal{A}$, hence $\mathcal{A} = \mathcal{A}^*$.

Let now $\mathbf{u}$ be a critical solution for the system. Then $v := \max(u_i, \ i \in \{1, \ldots, m\})$ is a critical subsolution for H. Moreover, as

$$\mathbf{u}(x) = u_1(x)\mathbb{1} \quad \text{for every } x \in \mathcal{A},$$

we deduce that $v = u_1$ on $\mathcal{A}$. Since $\mathcal{A} = \mathcal{A}^*$, there exists a critical solution $\tilde{u}$ for H such that $\tilde{u} = v$ on $\mathcal{A}$. Now the function $\tilde{\mathbf{u}} = \tilde{u}\mathbb{1}$ is a critical solution of the weakly coupled system satisfying $\tilde{\mathbf{u}} = \mathbf{u}$ on $\mathcal{A}$. By the comparison principle, i.e. Theorem 5.3, we conclude that $\mathbf{u} = \tilde{\mathbf{u}}$. □

Appendix A.

In this appendix we want to give a proof of Proposition 2.7.

In what follows, a function u will be said to be *semiconcave* on an open subset U of either $\mathbb{T}^N$ or $\mathbb{T}^N \times \mathbb{R}_+$ if, for every $x \in U$, there exists a vector $p_x \in \mathbb{R}^N$ such that

$$u(y) - u(x) \leqslant \langle p_x, y - x \rangle + |y - x|\, \omega(|y - x|) \qquad \text{for every } y \in U,$$

where ω is a modulus. The vectors p_x satisfying such inequality are precisely the elements of $D^+u(x)$, which is thus always nonempty in U. Moreover, $\partial_c u(x) = D^+u(x)$ for every $x \in U$. By the upper semicontinuity of the map $x \mapsto \partial_c u(x)$ with respect to set inclusion, we get in particular that Du is continuous in its domain of definition, see [6].

Next, we prove the following

Proposition A.1. *Let $T > 0$ and $G : [0, T] \times \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz Hamiltonian such that $G(s, \cdot, \cdot)$ is a strictly convex Hamiltonian, for every fixed $s \in [0, T]$. Let $u(t, x)$ be a Lipschitz function in $[0, T] \times \mathbb{T}^N$ that solves the evolutive Hamilton-Jacobi equation*

$$\frac{\partial u}{\partial t} + G(t, x, D_x u) = 0 \qquad \text{in } (0, T) \times \mathbb{T}^N, \tag{32}$$

in the viscosity sense. Then

(i) for every $0 < \tau < T$, the function u is semiconcave in $[\tau, T) \times \mathbb{T}^N$;

(ii) if $u(0, \cdot)$ is semi–concave in $\mathbb{T}^N$, then the functions $\{\, u(t, \cdot) \,:\, t \in [0, T) \,\}$ are equi–semiconcave.

Proof. Since u is Lipschitz, up to modifying G outside $[0, T] \times \mathbb{T}^N \times B_R$ for a sufficiently large $R > 0$, we can assume that G is superlinear in p, uniformly with respect to (t, x). We are then in the setting considered by Cannarsa and Soner in [7] and item *(i)* follows from their results.

Let us prove *(ii)*. Let us denote by $L(t, x, q)$ the the Lagrangian associated with G through the Fenchel transform and by u_0 the initial datum $u(0, \cdot)$. It is well known, see for instance [6], that the following representation formula holds:

$$u(t, x) = \inf_{\xi(t)=x} \Big(u_0\big(\xi(0)\big) + \int_0^t L\big(s, \xi(s), \dot{\xi}(s)\big) \mathrm{d}s \Big), \qquad (t, x) \in (0, T) \times \mathbb{T}^N, \quad (33)$$

where the infimum is taken by letting ξ vary in the family of absolutely continuous curves from $[0, t]$ to $\mathbb{T}^N$. Moreover, the minimum is attained by some curve γ, which is, in addition, Lipschitz continuous (actually, of class C^1), see [9].

We claim that there exists a constant κ, only depending on G and on the Lipschitz constant of u in $[0, T] \times \mathbb{T}^N$, such that $\|\dot{\gamma}\|_\infty \leqslant \kappa$. To this aim, we apply Proposition 2.4 in [22] to the function $u(t, x)$ and the curve $s \mapsto (s, \gamma(s))$ to get

$$\frac{\mathrm{d}}{\mathrm{d}s} u\big(s, \gamma(s)\big) = p_t(s) + \langle p_x(s), \dot{\gamma}(s) \rangle \qquad \text{for a.e. } s \in [0, t], \quad (34)$$

where $s \mapsto \big(p_t(s), p_x(s)\big)$ is a measurable and essentially bounded function on $[0, t]$ such that

$$\big(p_t(s), p_x(s)\big) \in \partial_c u\big(s, \xi(s)\big) \qquad \text{for a.e. } s \in [0, t].$$

By integrating (34) and using the Fenchel inequality we get

$$\begin{aligned} u(t, x) &= u_0\big(\gamma(0)\big) + \int_0^t p_t(s) + \langle p_x(s), \dot{\gamma}(s) \rangle \,\mathrm{d}s \\ &\leqslant u_0\big(\gamma(0)\big) + \int_0^t p_t(s) + G\big(s, \gamma(s), p_x(s)\big) + L\big(s, \gamma(s), \dot{\gamma}(s)\big) \,\mathrm{d}s \\ &\leqslant u_0\big(\gamma(0)\big) + \int_0^t L\big(s, \gamma(s), \dot{\gamma}(s)\big) \,\mathrm{d}s, \end{aligned}$$

where in the last inequality we used the fact that u is a (sub)–solution of the time dependent equation, i.e.

$$p_t + G(t, x, p) \leqslant 0 \qquad \text{for every } (p_t, p_x) \in \partial_c u(t, x) \text{ and } (t, x) \in (0, T) \times \mathbb{T}^N.$$

Since γ is minimizing, all the inequalities must be equalities, in particular we obtain

$$\dot{\gamma}(s) \in \partial_p G\big(s, \gamma(s), p_x(s)\big) \qquad \text{for a.e. } s \in [0, t]. \quad (35)$$

This proves the claim by choosing

$$\kappa := \sup \big\{ |q| \,:\, q \in \partial_p G(s, x, p),\ (s, x) \in [0, T] \times \mathbb{T}^N,\ |p| \leqslant \mathrm{Lip}\,\big(u; [0, T] \times \mathbb{T}^N\big) \ \big\},$$

which is finite thanks to the convexity and the growth assumptions assumed on G with respect to p.

Let us now fix $t \in (0, T)$, $x_1, x_2 \in \mathbb{T}^N$, $\lambda \in [0, 1]$ and set $x = \lambda x_1 + (1 - \lambda) x_2$. Note that $x_1 = x + (1 - \lambda) h$ and $x_2 = x - \lambda h$ for $h = x_1 - x_2$. Let us denote by γ a curve realizing the infimum in (33) for such a pair of (t, x), by K a Lipschitz constant for

L restricted to $[0,T]\times\mathbb{T}^N\times B(0,2\kappa)$ and by ω a semi–concavity modulus for u_0. We get

$$\begin{aligned}
&\lambda u(t,x_1)+(1-\lambda)u(t,x_2)-u(t,x)\\
&\qquad\leqslant \lambda\Big(u_0\big(\gamma(0)+(1-\lambda)h\big)+\int_0^t L\big(s,\gamma(s)+(1-\lambda)h,\dot\gamma(s)\big)\mathrm{d}s\Big)\\
&\qquad\quad+(1-\lambda)\Big(u_0\big(\gamma(0)-\lambda h\big)+\int_0^t L\big(s,\gamma(s)-\lambda h,\dot\gamma(s)\big)\mathrm{d}s\Big)\\
&\qquad\quad-\Big(u_0\big(\gamma(0)\big)+\int_0^t L\big(s,\gamma(s),\dot\gamma(s)\big)\mathrm{d}s\Big)\\
&\qquad=\lambda u_0\big(\gamma(0)+(1-\lambda)h\big)+(1-\lambda)u_0\big(\gamma(0)-\lambda h\big)-u_0\big(\gamma(0)\big)\\
&\qquad\quad+\lambda\Big(\int_0^t\Big(L\big(s,\gamma(s)+(1-\lambda)h,\dot\gamma(s)\big)-L\big(s,\gamma(s),\dot\gamma(s)\big)\Big)\mathrm{d}s\\
&\qquad\quad+(1-\lambda)\int_0^t\Big(L\big(s,\gamma(s)-\lambda h,\dot\gamma(s)\big)-L\big(s,\gamma(s),\dot\gamma(s)\big)\Big)\mathrm{d}s\\
&\qquad\leqslant\lambda(1-\lambda)\Big(\omega(|x_1-x_2|)+t\,K|x_1-x_2|\Big),
\end{aligned}$$

which proves the assertion. □

The result just proved will be applied to weakly coupled systems as follows:

Proposition A.2. *Let $T>0$ and $\mathbf{u}=(u_1,\dots,u_m)\in\big(\mathrm{Lip}([0,T]\times\mathbb{T}^N)\big)^m$ be a solution of the evolutionary weakly coupled system (7). Let $i\in\{1,\dots,m\}$ and suppose that H_i is locally Lipschitz and strictly convex. Then, for all $0<\tau<T$, the function u_i restricted to $[\tau,T)\times\mathbb{T}^N$ is semiconcave. Moreover, if, the initial condition $u_i(0,\cdot)$ is semiconcave, then the functions $\{\,u_i(t,\cdot)\,:\,t\in[0,T]\,\}$ are equi–semiconcave.*

Proof. The function u_i solves, for the given index $i\in\{1,\dots,m\}$, a Hamilton–Jacobi equation of the kind (32) with

$$G(t,x,p)=H_i(x,p)+\big(B(x)\mathbf{u}(t,x)\big)_i,\qquad (t,x,p)\in[0,T]\times\mathbb{T}^N\times\mathbb{R}^N.$$

The conclusion follows by applying Proposition A.1. □

We are now ready to prove Proposition 2.7.

Proof of Proposition 2.7. We recall that, by convexity of the Hamiltonians, subsolutions to the critical system coincide with almost everywhere subsolutions. This fact will be repeatedly exploited along the proof.

Assume first that $t\mapsto\mathcal{S}(t)\mathbf{u}+t\,a\mathbb{1}$ is non–decreasing. Pick $t_0>0$ such that the map $(t,x)\mapsto\mathcal{S}(t)\mathbf{u}(x)$ is differentiable at (t_0,x) for almost every $x\in\mathbb{T}^N$ and

$$\partial_t\mathcal{S}(t_0)\mathbf{u}(x)\geqslant -a\qquad\text{for a.e. } x\in\mathbb{T}^N.$$

By the Lipschitz character of the map $(t,x)\mapsto\mathcal{S}(t)\mathbf{u}(x)$ and Fubini's theorem, this holds true for almost every $t_0>0$. Using the evolutionary equation, which is verified at every differentiability point of $\mathcal{S}(t)\mathbf{u}(x)$, we deduce that, for every $i\in\{1,\dots,m\}$,

$$H_i\Big(x,D\big(\mathcal{S}(t_0)\mathbf{u}\big)_i(x)\Big)+\big(B(x)\mathcal{S}(t_0)\mathbf{u}(x)\big)_i\leqslant a\quad\text{for a.e. } x\in\mathbb{T}^N,$$

that is, $\mathcal{S}(t_0)\mathbf{u} \in \mathcal{H}(a)$. This being true for almost every $t_0 > 0$, the conclusion follows by stability of viscosity subsolutions.

Let us now assume reciprocally that $\mathbf{u} \in \mathcal{H}(a)$. We first approximate each Hamiltonian H_i with a sequence $(H_i^k)_k$ of convex Hamiltonians that are, in addition, locally Lipschitz in (x,p) and strictly convex in p. This can be done by taking a sequence $(\rho_k)_k$ of standard mollifiers on $\mathbb{R}^N$ and by setting

$$H_i^k(x,p) = \int_{B_1} \rho_k(y) H_i(x-y,p)\,\mathrm{d}y + \frac{|p|^2}{k}, \qquad (x,p) \in \mathbb{T}^N \times \mathbb{R}^N.$$

Note that, for each index $i \in \{1,\dots,m\}$, $\quad H_k^i \rightrightarrows H^i \quad$ in $\mathbb{T}^N \times \mathbb{R}^N$ as $k \to +\infty$. Let us denote by $\mathcal{H}_k(a)$ the set of $\mathbf{a}$–subsolution of the weakly coupled system (6) with $\mathbf{a} = a\mathbb{1}$ and $H_1^k, \dots, H_m^k$ in place of $H_1, \dots, H_m$, and by $\mathcal{S}_k$ the semigroup associated with the corresponding time–dependent equation (7).

Next, we approximate $\mathbf{u}$ with a sequence of $(\mathbf{u}^n)_n$ of functions that are component–wise semi–concave by setting

$$u_i^n(x) = \inf_{y \in \mathbb{T}^N} u_i(y) + n|y-x|^2 \qquad \text{for every } x \in \mathbb{T}^N \text{ and } i = 1, \cdots, m.$$

Fix $\varepsilon > 0$. A standard argument shows that, for n large enough, $\mathbf{u}^n \in \mathcal{H}(a+\varepsilon)$. Moreover, by the Lipschitz character of $\mathbf{u}^n$ and by the local uniform convergence of $(H_1^k, \dots, H_m^k)$ to $(H_1, \dots, H_m)$, we also have that $\mathbf{u}^n \in \mathcal{H}_k(a+2\varepsilon)$ for k sufficiently large. We now apply Proposition A.2 to infer that the map $(t,x) \mapsto \mathcal{S}(t)\mathbf{u}^n(x)$ is semiconcave in $[0,\tau] \times \mathbb{T}^N$ for every $\tau > 0$. By using the fact that the gradient of a semiconcave function is continuous in its domain of definition and by choosing $\tau > 0$ small enough, we get $\mathcal{S}(t)\mathbf{u}^n \in \mathcal{H}_k(a+3\varepsilon)$ for every $0 \leqslant t \leqslant \tau$. By exploiting this information in the evolutive weakly coupled system, we get

$$\frac{\partial}{\partial t}\mathcal{S}_k(t)\mathbf{u}^n(x) \geqslant -(a+3\varepsilon)\mathbb{1} \qquad \text{for a.e. } (t,x) \in (0,\tau) \times \mathbb{T}^N,$$

i.e.

$$\mathcal{S}_k(t+h)\mathbf{u}^n \geqslant \mathcal{S}_k(t)\mathbf{u}^n - h(a+3\varepsilon)\mathbb{1} \qquad \text{for every } 0 < t < t+h \leqslant \tau.$$

Now, by the comparison principle for the evolution equation and by using the fact that the semigroup commutes with the addition of scalar multiples of the vector $\mathbb{1}$, we obtain that $t \mapsto \mathcal{S}_k(t)\mathbf{u}^n - t(a+3\varepsilon)\mathbb{1}$ is non decreasing. We now exploit the fact that

$$\mathcal{S}_k(t)\mathbf{u}^n \underset{k\to+\infty}{\rightrightarrows} \mathcal{S}(t)\mathbf{u}^n \quad \text{and} \quad \mathcal{S}(t)\mathbf{u}^n \underset{n\to+\infty}{\rightrightarrows} \mathcal{S}(t)\mathbf{u} \qquad \text{in } \mathbb{R}_+ \times \mathbb{T}^N$$

to infer that

$$t \mapsto \mathcal{S}(t)\mathbf{u}^n - t(a+3\varepsilon)\mathbb{1} \quad \text{is non–decreasing on } [0,+\infty).$$

Being this true for every $\varepsilon > 0$, we finally have that $t \mapsto \mathcal{S}(t)\mathbf{u}^n - ta\mathbb{1}$ is non–decreasing on $[0,+\infty)$.

The last assertion follows from the equivalence just proved, together with the fact that the semigroup $\mathcal{S}(t)$ is non–decreasing and commutes with addition of vectors of the form $a\,\mathbb{1}$ with $a \in \mathbb{R}$. $\square$

The next result can be seen as a refinement of Proposition 2.7, where we assume inequality to hold only for one of the equations of the system. The proof follows via the same argument and is omitted.

Proposition A.3. *Let* $\mathbf{u} \in \big(\mathrm{Lip}(\mathbb{T}^N)\big)^m$ *and assume that there exist an index* $i \in \{1, \dots, m\}$*, an open subset* U *of* $\mathbb{T}^N$ *and* $a \in \mathbb{R}$ *such that*

$$H_i\big(x, Du_i(x)\big) + \big(B(x)\mathbf{u}(x)\big)_i \leqslant a \qquad \text{for a.e. } x \in U.$$

Then for every open set V *compactly contained in* U *there exists* $\tau_V > 0$ *such that*

$$\big(\mathcal{S}(t)\mathbf{u}\big)_i(x) \geqslant u_i(x) - a\,t \qquad \text{for every } x \in V \text{ and } t \in [0, \tau_V].$$

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